Difference Between Spin and Orbital Angular Momentum

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A paraxial circularly polarized light beam and a radiation of a rotating electric dipole are considered. Reasons are presented that the rotating mass-energy of these electromagnetic fields represents orbital angular momentum rather than spin angular momentum. On the other hand, a Feynman's quantum mechanics calculation of spin is presented for the radiation of a rotating electric dipole. This calculation shows that the rotating mass-energy of the radiation is not connected with the spin because, in particular, the moment of momentum is emitted mainly into the equatorial part of such a radiation, whereas spin is emitted into polar regions. An attempt is made to obtain the Feynman's result in the frame of classical field theory.

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A circularly polarized light beam carries an angular momentum (AM) [1,2]. However, irritative questions exist: what is the distribution of this AM over the beam section, and what is the nature of this AM, orbital or spin? The first question was discussed in [3]. Here we consider two examples, which help to answer the second question.

1. Angular momentum of a light beam

A paraxial circularly polarized Laguerre-Gaussian beam [4], LG_p^l, in the cylindrical coordinates ρ, φ, z with the metric $dl^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$, namely

$$\underset{\rightarrow}{E} = \exp\{i(l+1)\varphi + i\omega(z-t)\}(\omega \underset{\rightarrow}{\rho} + i\omega\rho \underset{\rightarrow}{\varphi} + i \underset{\rightarrow}{z} \partial_\rho) u_p^l(\rho, z), \quad \underset{\rightarrow}{B} = -i \underset{\rightarrow}{E},$$

$$u_p^l = \frac{C_p^l}{w(z)}\left[\left(\frac{\rho\sqrt{2}}{w}\right)^l L_p^l\left(\frac{2\rho^2}{w^2}\right)\right]\exp\left\{-\frac{\rho^2}{w^2} + \frac{i\rho^2}{w^2 z_R} - i(2p+l+1)\arctan\left(\frac{z}{z_R}\right)\right\} \tag{1}$$

($\underset{\rightarrow}{\rho}, \underset{\rightarrow}{\varphi}, \underset{\rightarrow}{z}$ are *covariant* coordinate vectors, $k = \omega$, $c = 1$) is an eigenfunction of the *orbital*, not spin, AM operator $-i\hbar\partial_\varphi$ with the eigenvalue $\hbar(l+1)$. This means that both, the circular polarization and the spiral phase front related with $l > 0$, carry only orbital AM, not spin, in the frame of the standard electrodynamics. This contradicts the opinion that the circulating energy flow represents the spin of the beam (see, e.g. [2]).

2. Radiation of a rotating electric dipole

Now we consider an exact, not paraxial, solution of the Maxwell equations; the solution for the radiation of a rotating electric dipole [5-7] in the spherical coordinates r, θ, φ:

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$$E^r = (2/r^3 - i2\omega/r^2)\sin\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi\,, \tag{2}$$

$$E^\theta = (-1/r^4 + i\omega/r^3 + \omega^2/r^2)\cos\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi, \tag{3}$$

$$E^\varphi = (-i/r^4 - \omega/r^3 + i\omega^2/r^2)\exp[i\varphi + i\omega(r-t)]/(4\pi\sin\theta), \tag{4}$$

$$B_{r\theta} = (i\omega/r + \omega^2)\cos\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi, \tag{5}$$

$$B_{\varphi r} = (\omega/r - i\omega^2)\sin\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi, \quad B_{\theta\varphi} = 0. \tag{6}$$

The angular distribution of the energy flux,

$$dP/d\Omega = <(\mathbf{E}\times\mathbf{B})_r r^2> = \omega^4(1+\cos^2\theta)/(32\pi^2)\,, \tag{7}$$

is depicted in Fig. 1 from [5]. The angular distribution of z-component of the moment of momentum flux, i.e., of torque,

$$dL_z/dtd\Omega = d\tau_z/d\Omega = <[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_z r^2> = \omega^3\sin^2\theta/(16\pi^2)\,, \tag{8}$$

is depicted in Fig. 2. The total power and total torque are

$$P = \int \omega^4(1+\cos^2\theta)/(32\pi^2)\sin\theta d\theta d\varphi = \omega^4/6\pi\,, \quad \tau_z = \int \omega^3\sin^2\theta/(16\pi^2)\sin\theta d\theta d\varphi = \omega^3/6\pi\,. \tag{9}$$

Note the ratio

$$\tau_z/P = 1/\omega\,. \tag{10}$$

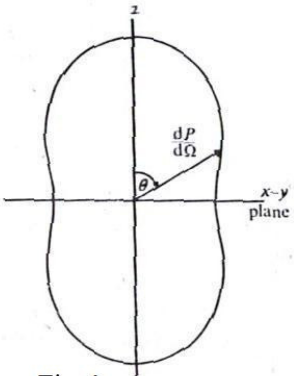

Fig. 1.

Angular distribution of the energy flux.

$dP/d\Omega \propto (\cos^2\theta + 1)$

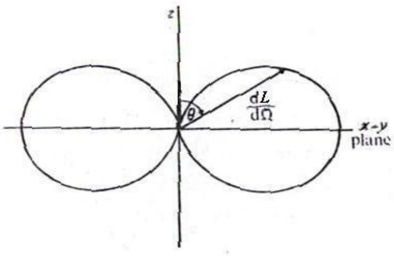

Fig. 2.

Angular distribution of z-component of the moment of momentum flux

$dL_z/dtd\Omega \propto \sin^2\theta$

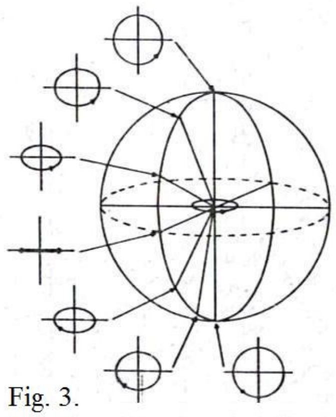

Fig. 3.

Polarization of the electric field seen by looking from different directions at a circular oscillator

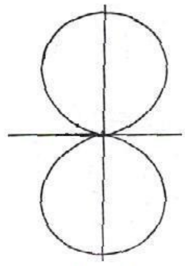

Fig. 4.

Degree of the circular polarization

$\sigma \cong \cos\theta$

We present also a distribution of the degree of circular polarization σ of the radiation [5], which approximately equals the ratio of lengths of the axes of the ellipse: $\sigma \cong \cos\theta$ (Fig. 3 and 4).

It is seen that orbital AM (8) is emitted mainly into the equatorial part of space, situated near the $x-y$-plane where the polarization is elliptic or linear. Polar regions, situated near the z-axis, are scanty by AM (8) although they are intensively illuminated by the almost circularly polarized radiation. So, if we associate spin of an electromagnetic radiation with a circular polarization, we must recognize AM (8) is an *orbital* AM, not spin. Also note, fields (2) – (6) are eigenfunctions of the *orbital*, not spin, AM operator, $-i\hbar\partial_\varphi$, with eigenvalue $\hbar$. This confirms the orbital nature of AM (8).

Note also that torque $\tau_z = P/\omega$ of Eqs. (8) - (10) cannot be provided by spin because the ratio spin/energy for a photon is $S/W = 1/\omega$, and for the z-component, S_z, the ratio provided by spin must be less, $S_z/W = \tau_{\text{spin}\,z}/P < 1/\omega$ contrary to Eq (10) [see Eq (18)].

3. Spin flux density of the dipole radiation

A simple calculation of *spin* flux density of the radiation was given by Feynman [8]. But his calculation is beyond the standard electrodynamics. Really, the *amplitudes* that a RHC photon and a LHC photon are emitted in the direction θ are (18.1), (18.2)

$$a(1+\cos\theta)/2 \quad \text{and} \quad -a(1-\cos\theta)/2. \tag{11}$$

So, in the direction, the spin flux density is proportional to

$$[a(1+\cos\theta)/2]^2 - [a(1-\cos\theta)/2]^2 = a^2\cos\theta. \tag{12}$$

It may compare with Fig. 4. The projection the spin flux density on z-axis is

$$dS_z / dtd\Omega \propto a^2\cos^2\theta. \tag{13}$$

At the same time, expressions (11) give the power density (7), Fig. 1:

$$dP/d\Omega \propto [a(1+\cos\theta)/2]^2 + [a(1-\cos\theta)/2]^2 = a^2(1+\cos^2\theta)/2. \tag{14}$$

4. Classical electrodynamics spin

The result (13) was obtained [6,7] by the use of a spin tensor [9-11]

$$\Upsilon^{\lambda\mu\nu} = A^{[\lambda}\partial^{|\nu|}A^{\mu]} + \Pi^{[\lambda}\partial^{|\nu|}\Pi^{\mu]} \tag{15}$$

in the frame of a modified classical electrodynamics. Here A^λ and Π^λ are the magnetic and electric vector potentials which satisfy $\partial_\lambda A^\lambda = \partial_\lambda \Pi^\lambda = 0$, $2\partial_{[\mu}A_{\nu]} = F_{\mu\nu}$, $2\partial_{[\mu}\Pi_{\nu]} = -e_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$, where $F^{\alpha\beta} = -F^{\beta\alpha}$, $F_{\mu\nu} = F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ is the field strength tensor of a free electromagnetic field; $e_{\mu\nu\alpha\beta}$ is the Levi-Civita antisymmetric tensor density.

Using (15) yields an angular distribution of z-component of the spin flux in the rotating dipole radiation,

$$dS_z / dtd\Omega = \omega^3\cos^2\theta/(16\pi^2), \tag{16}$$

and the total z-component of the spin flux,

$$dS_z / dt = \int \omega^3\cos^2\theta/(16\pi^2)\sin\theta d\theta d\varphi = \omega^3/(12\pi), \tag{17}$$

which is half of the total orbital angular momentum flux (9). So, instead of (10) we have the ratio

$$dS_z /(dt\, P) = 1/(2\omega) \tag{18}$$

as it must be for spin. However, the ratio of the spin flux density (16) to the power density (7) at $\theta = 0$ equals $1/\omega$,

$$\left.\frac{dS_z}{dt\,dP}\right|_{\theta=0} = \left.\frac{\omega^3\cos^2\theta/(16\pi^2)}{\omega^4(1+\cos^2\theta)/(32\pi^2)}\right|_{\theta=0} = \frac{1}{\omega}, \tag{19}$$

just as for a photon because the radiation is circularly polarized with plane phase front along z-axis.

Imagine our rotating dipole is surrounded by an absorbing sphere. Then different photons carry different z- component of AM to the sphere. Putting together (8) and (16) yields

$$\frac{dJ_z}{dtdP} = \frac{dL_z + dS_z}{dtdP} = \frac{2}{\omega(\cos^2\theta+1)}. \tag{20}$$

This means, if the wave function of a photon collapses in a pole of the sphere $(\theta = 0)$, the pole catchs pure spin of $\hbar$, and if the wave function collapses in a point at equator of the sphere $(\theta = \pi/2)$, the point catchs, on average, a pure orbital AM because the photon is plane polarized. And this orbital AM equals $2\hbar$. On average, a photon carries AM of $3\hbar/2$.

Conclusions and Acknowledgements

We must recognize the standard electrodynamics does not catch sight of spin of electromagnetic fields. The proposed spin tensor describes spin of photons in the frame of the modified classical electrodynamics.

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