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CLASSICAL SPIN IN SPACE WITH AND WITHOUT TORSION

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Operational definitions of the energy-momentum and spin tensors and the equations connecting these tensors in curved space and in space with torsion are considered. According to such definitions, these tensors uniquely localize the momentum and spin. By the definition of the spin tensor, as an example, absorption of spin of a circularly polarized electromagnetic wave is calculated for an electro-conducting medium. Such absorption results in an asymmetric energy-momentum tensor of the medium. The widespread objections against the reality of classical spin tensors are considered and rejected. A useless nature of the Belinfante symmetrization procedure is shown. The derivation of the metric energy-momentum tensors is criticized. Attention is directed to an analogy between the momentum and angular momentum conservation laws, on the one hand, and the equations of motion of a dipole particle on the other.

Классический спин в пространстве с кручением и без кручения
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Рассматриваются операционные определения классических тензоров энергии-импульса и спина, а также уравнения, связывающие их в искривленном пространстве с кручением и без кручения. Согласно таким определениям, эти тензоры однозначно локализуют энергию, импульс и момент импульса, как орбитальный, так и спиновый. В качестве примера определение тензора спина использовано для подсчета поглощения спина электромагнитной волны электропроводящей средой. Такое поглощение создает механическое состояние среды, характеризующееся несимметричным тензором энергии-импульса. Показана бесполезность процедуры Белинфанте, симметризирующей канонический тензор энергии-импульса. Критикуется вывод метрического тензора энергии-импульса. Обращается внимание на аналогию между законами сохранения энергии-импульса и момента импульса, с одной стороны, и уравнениями движения дипольной частицы, с другой.

From the beginning of the 1960s, Prof. D.D. Ivanenko repeatedly supported the use of Cartan's torsion of space-time in physics. The problem of torsion is now far from solution as well. This problem is considered in the present article.

1. Energy-momentum tensor

In electrodynamics, the 4-current density j^α is a source of an electromagnetic field expressed by the electromagnetic field tensor $F^{\alpha\beta}$:

$$\partial_\beta F^{\alpha\beta} = -j^\alpha.$$

Similarly, in Einstein's general relativity, the energy-momentum tensor $T^{\alpha\beta}$ is a source of the gravitational field which is expressed either by the curvature tensor $R_{\mu\nu\alpha}^{\cdot\cdot\cdot\beta}$ or by the connection coefficients $\Gamma^\alpha_{\mu\nu}$, or by the metric tensor $g_{\mu\nu}$ as functions of the space-time coordinates,

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - g^{\alpha\beta}R/2 = \kappa T^{\alpha\beta}.$$

The energy-momentum tensor is the "current" which "produces" gravitation. And we agree with [1], that it ought to be an observable quantity of fundamental importance, which does not admit ambiguity.

Hermann Weyl recognized [2]:

> 'The general theory of relativity alone, which allows the process of variation to be applied to the metrical structure of the world, leads to a true definition of energy'.

However, the observability of the energy-momentum tensor has also a more direct expression, which has not been connected to gravitation. Synge wrote [3]:

> 'Following the suggestion of the statistical model, we assign to a material continuum a *symmetric energy tensor* ...It embodies the mechanical properties of matter, such as stress and density ...We borrow from the statistical model the interpretation of the energy tensor in terms of fluxes, and we make the following statement: (flux of 4-momentum across a 3-target dV) $= T^{\alpha\beta}dV_\beta$.'

Thus there is an operational definition of the energy-momentum tensor: If a substance and (or) a field are

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locally bounded by an infinitesimal element dV_β, this element receives the infinitesimal 4-momentum

$$dP^\alpha = T^{\alpha\beta} dV_\beta. \tag{1.1}$$

The spatial part of the energy-momentum tensor $T^{\alpha\beta}$, the tensor T^{ik}, is referred to as the stress tensor. $T^{ik}da_k$ is the i-th component of the force acting on the surface element da_k [4, p. 15]:

$$d\mathcal{F}^i = T^{ik} da_k. \tag{1.2}$$

Therefore T^{ik} is the momentum flux density.

Integration of the expression (1.1) over a closed hypersurface in Minkowski space gives the 4-momentum transferred to the internal side of the boundary $V = \partial\Omega$ of the 4-volume Ω, that is, the 4-momentum received by matter from external sources in the 4-volume Ω:

$$P^\alpha = \oint T^{\alpha\beta} dV_\beta = \int \partial_\beta T^{\alpha\beta} d\Omega.$$

It means that the divergence $\partial_\beta T^{\alpha\beta}$ equals to the 4-momentum, which is formed in a unit 4-volume due to external sources; in other words, it is the external 4-force density,

$$f^\alpha = \partial_\alpha T^{\alpha\beta},$$

which acts on the matter from external objects and is transferred to the hypersurface through the matter. For example, if the matter is an electromagnetic field with Maxwell's energy-momentum tensor

$$\underset{e}{T}{}^{\alpha\beta} = -F^\alpha{}_\nu F^{\beta\nu} + g^{\alpha\beta} F_{\mu\nu} F^{\mu\nu}/4, \tag{1.3}$$

then the divergence of the energy-momentum tensor is the Lorentz force with an inverse sign:

$$\partial_\beta \underset{e}{T}{}^{\alpha\beta} = -F^{\alpha\nu} j_\nu,$$

because the Lorentz force $F^{\alpha\nu}j_\nu$ acts on currents due to a field, and $-F^{\alpha\nu}j_\nu$ acts on the field due to currents, which are external objects in relation to the field. If

$$\partial_\beta T^{\alpha\beta} = 0, \tag{1.4}$$

then the matter is in equilibrium with respect to the momentum. We say that the matter is closed with respect to momentum.

2. Spin tensor

Is the energy-momentum tensor symmetric? Apparently, the energy-momentum tensor of the electromagnetic field (1.3) is symmetric. However, it is necessary to take into account other opportunities. We shall consider a certain elastic medium, dielectric or conducting electricity. Let this medium absorb a flow of electrons (photons, neutrinos) which has a circular polarization. It is obvious that as the flow is absorbed, the spin angular momentum brought by it is also absorbed by the medium. Therefore, this medium experiences a rotating mechanical action allocated in a volume. As a result, any volume of this medium acts by a torque on its border located inside the medium or at its edge:

$$\tau^{ij} = 2 \oint r^{[i} T^{j]k} da_k.$$

Using the Stokes theorem yields

$$\tau^{ij} = 2 \oint r^{[i} T^{j]k} da_k = 2 \int \partial_k (r^{[i} T^{j]k}) dV = 2 \int (-T^{[ij]} + r^{[i} f^{j]}) dV.$$

Due to uniformity of the absorbed spin, the second term on the right is easily eliminated by choosing a suitable point in the middle of the volume V as the origin of the radius-vector. However, the first term containing an antisymmetric part of the stress tensor characterizes a specific mechanical condition of medium, which arises at spin absorption.

Consider this condition of the medium. Let

$$T^{yx} = -T^{xy} = 1. \tag{2.1}$$

Let us pick out a unit cube in the medium, with sides perpendicular to the axes. The force $d\mathcal{F}^y = T^{yx}da_x = 1$ acts on the adjacent medium at the side with the covariant coordinates $(da_x = 1, 0, 0)$, while the force $d\mathcal{F}^x = T^{xy}da_y = -1$ acts on the adjacent medium at the side with the covariant coordinates $(0, da_y = 1, 0)$. As a result, both forces try to rotate the medium near the cube in the positive direction around the z axis. Therefore the medium experiences the torque

$$d\tau^{xy} = x d\mathcal{F}^y - y d\mathcal{F}^x = 2 = -2T^{[xy]}.$$

The torque increases with the cube size. This expresses the fact that $-2T^{[xy]}$ is the divergence of the moment of momentum flux in accordance with the equality

$$2\partial_k (r^{[i} T^{j]k}) = -2T^{[ij]}.$$

Naturally, this condition is due to absorption of an external spin flow by the medium. We shall use a special designation for the spin flux density, $\Upsilon^{ijk} = \Upsilon^{[ij]k}$. It means that the spin angular momentum, which flows through an element da_k per unit time, i.e. the spin flux, i.e. torque, is

$$\tau^{ij} = \Upsilon^{ijk} da_k. \tag{2.2}$$

Therefore, the angular momentum conservation law requires

$$-2T^{[ij]} = -\partial_k \Upsilon^{ijk}.$$

Let us carry out a similar reasoning for the 4-spin tensor $\Upsilon^{\alpha\beta\kappa}$. The quantity

$$J^{\alpha\beta} = \oint (2r^{[\alpha} T^{\beta]\kappa} + \Upsilon^{\alpha\beta\kappa}) dV_\kappa = \int (-2T^{[\alpha\beta]} + 2r^{[\alpha} \partial_\kappa T^{\alpha]\kappa} + \partial_\kappa \Upsilon^{\alpha\beta\kappa}) d\Omega$$

represents the full angular momentum (orbital + spin), gained by the medium in the 4-volume Ω in case the pair $(T^{\alpha\kappa},\ \Upsilon^{\alpha\beta\kappa})$ is inherent to the medium. We say that the medium is closed or is in equilibrium with respect to the spin if

$$\partial_\kappa \Upsilon^{\alpha\beta\kappa} - 2T^{[\alpha\beta]} = 0. \tag{2.3}$$

3. An example of spin absorption by electro-conducting medium

As an example, consider an electromagnetic wave with circularly polarization [5],

$$\begin{aligned} &\breve{\mathbf{E}} = \exp[i(\breve{k}z - t)](\mathbf{x} + i\mathbf{y}), && \breve{\mathbf{B}} = -i\breve{k}\breve{\mathbf{E}},\\ &\breve{k} = \sqrt{\breve{\epsilon}} = k' + ik'', && \breve{\epsilon} = 1 + i\gamma,\\ &k'^2 - k''^2 = 1, && 2k'k'' = \gamma,\\ &2k'^2 = k^2 + 1, && 2k'' = k^2 - 1, \end{aligned}$$

which moves in an electro-conductor at $z > 0$. Here k, ϵ, γ are the wave number, permittivity and conductivity, respectively. The symbol "breve" marks complex vectors and numbers. For brevity, we put the speed of light $c = 1$ and the frequency $\omega = 1$.

In space at $z < 0$, there exist an incident wave,

$$\breve{\mathbf{E}}_1 = (1 + \breve{k})\,\mathrm{e}^{i(z-t)}(\mathbf{x} + i\mathbf{y})/2, \qquad \breve{\mathbf{B}}_1 = -i\breve{\mathbf{E}}_1$$

and an outgoing wave,

$$\breve{\mathbf{E}}_2 = (1 - \breve{k})\,\mathrm{e}^{i(-z-t)}(\mathbf{x} + i\mathbf{y})/2, \qquad \breve{\mathbf{B}}_2 = i\breve{\mathbf{E}}_2.$$

Let us find the spin flux in space, using an expression of the electrodynamics spin tensor from the article submitted to JETP on January 27, 1999 (see also [6]):

$$\underset{\mathrm{em}}{\Upsilon}{}^{xyz} = A^{[x}\partial^{|z|}A^{y]} + \Pi^{[x}\partial^{|z|}\Pi^{y]}, \qquad \partial^z = -\partial_z. \tag{3.1}$$

Here A^i and Π^i are the magnetic and electric vector potentials. At $z < 0$,

$$\begin{aligned} \breve{\mathbf{A}}_0 &= -\int(\breve{\mathbf{E}}_1 + \breve{\mathbf{E}}_2)dt = -i(\breve{\mathbf{E}}_1 + \breve{\mathbf{E}}_2),\\ \breve{\Pi}_0 &= -\int(\breve{\mathbf{B}}_1 + \breve{\mathbf{B}}_2)dt = \breve{\mathbf{E}}_1 - \breve{\mathbf{E}}_2. \end{aligned} \tag{3.2}$$

We calculate the average spin flux density. A short result is obtained:

$$\begin{aligned} <\underset{em}{\Upsilon}{}_0^{xyz}> &= \Re(\overline{A}_0{}^{[x}\partial^{|z|}\breve{A}_0{}^{y]} + \overline{\Pi}_0{}^{[x}\partial^{|z|}\breve{\Pi}_0{}^{y]})/2 = k',\\ k'^2 &= (1 + \sqrt{1+\gamma^2})/2. \end{aligned}$$

The overbar means complex conjugation.

The spin flux falling on the conductor from space is gradually absorbed in the conductor and is consequently a function of z. To calculate the spin flux in the conductor, it is impossible to apply Eq. (3.1) because the presence of electric currents deprives the electric vector potential of an unequivocal sense. It is necessary to use the formula

$$\underset{em}{\Upsilon}{}_0^{xyz} = 2A^{[x}\partial^{|z|}A^{y]}, \qquad \mathbf{A} = -\int \mathbf{E}dt$$

in a conductor; the formula was presented in a paper submitted to JETP Letters on May 14, 1998. This formula gives

$$\begin{aligned} \langle\, \underset{em}{\Upsilon}{}_0^{xyz} \,\rangle &= \Re(\overline{A}_0{}^{[x}\partial^{|z|}\breve{A}_0{}^{y]} = k' \exp(-2k''z).\\ -\,\partial_z\langle\, \underset{em}{\Upsilon}{}_0^{xyz} \,\rangle &= 2k'k''\exp(-2k''z) = \gamma\exp(-2k''z). \end{aligned}$$

The divergence of the spin tensor shows that the conductor experiences the torque density $d\tau^{xy}/dV$. The torque density creates a mechanical stress described by the antisymmetric part of the stress tensor in a rigid conductor:

$$\begin{aligned} \partial\tau^{xy}/dV &= -\partial_z \underset{em}{\Upsilon}{}^{xyz} = -2T^{[xy]},\\ T^{[xy]} &= \gamma\exp(-2k\text{''}z)/2 = -T^{[yx]}. \end{aligned}$$

Note that it is the condition of medium that was considered earlier, (2.1).

Thus an electromagnetic wave in a conductor, taken together with the conductor, form a closed matter system relative to spin in accordance with Eq. (2.3). Thus, as is easily checked, differentiation of Eq. (3.1) yields

$$d\tau^{xy}/dV = -\partial_z \underset{em}{\Upsilon}{}^{xyz} = 2j^{[x}A^{y]}.$$

So, $2j^{[x}A^{y]}$ is a rotating analogue of the Lorentz force. So, it has been found, that an electromagnetic field acts on currents not only by the force j_kF^{ik}, but also by the torque $2j^{[i}A^{k]}$.

4. Popular objections against the spin tensor

Despite the obvious reasons presented in Sec. 2 and 3, the spin tensor of electrodynamics (3.1) is not recognized and even not known. And, what is more, it is in a usual manner to deny the reality of the spin tensor in general [7]. The brightest objection against the spin tensor is the proclaimed symmetry of the energy-momentum tensor of matter [8]. In the popular monograph [9], one can read, 'All the stress-energy tensors explored above were symmetric. That they could not have been otherwise one sees as follows ...' (p. 141). There is no concept of spin tensor in [9]. The known asymmetry of the canonical energy-momentum tensor, which is, by the way, unified in a pair with the canonical spin tensor,

$$\left(\underset{c}{T}{}^{\alpha\beta} = \partial^\alpha A_\sigma \frac{\partial\Lambda}{\partial(\partial_\beta A_\sigma)} - g^{\alpha\beta}\Lambda,\right.$$
$$\left.\underset{c}{\Upsilon}{}^{\alpha\beta\kappa} = -2A^{[\alpha}\delta_\sigma^{\beta]}\frac{\partial\Lambda}{\partial(\partial_\kappa A_\sigma)}\right), \tag{4.1}$$

is considered as a reason for a violation of the angular momentum conservation law. The asymmetry is subject to elimination. On page 505 OF [9] one can read:

'Field theory provides a ... method of generating the so-called canonical expression for the stress-energy tensor of the field. By the very manner of construction, such an expression is guaranteed also to satisfy momentum and energy conservation law, and by this circumstance, it becomes useful in certain contexts. However, the canonical tensor is often not symmetric in its two indices, and in such cases violates the angular momentum conservation law. Even when symmetric, it may give a quite different localization of stress and energy than that given by (5.1) (*see below*). Field theory by itself is unable to decide between these different pictures of where the field energy is localized'.

Rosenfeld declared that the symmetry of the energy-momentum tensor is not an arbitrary choice, but is compellingly demanded by Einstein's theory of gravity, which is compatible with a symmetric energy-momentum tensor only [10].

Therefore all physicists try to symmetrize the canonical tensor by force using the Belinfante procedure [11] and to transform the canonical pair (4.1) into a pair, which, as it is considered, is equivalent to (4.1), but is made up of a symmetric energy-momentum tensor and a zero spin tensor:

$$\left(\underset{s}{T}{}^{\alpha\beta},\ 0\right).$$

A strange belief is laid in a basis of the well-known Belinfante procedure. It is believed that, contrary to such formulae as (1.1), (2.2), the physical content of the tensors comprising the pair $(T^{\alpha\beta},\ \Upsilon^{\alpha\beta\kappa})$ does not vary due to addition of an arbitrary pair $(\Delta T^{\alpha\beta},\ \Delta\Upsilon^{\alpha\beta\kappa})$, which satisfies the conservation laws (1.4), (2.3):

$$\partial_\beta T^{\alpha\beta}=0, \qquad \partial_\kappa\Upsilon^{\alpha\beta\kappa}-2T^{[\alpha\beta]}=0, \tag{4.2}$$

i.e.,

$$\partial_\beta(\Delta T^{\alpha\beta})=0, \qquad \partial_\kappa(\Delta\Upsilon^{\alpha\beta\kappa})-2\Delta T^{[\alpha\beta]}=0. \tag{4.3}$$

They believe that these tensors "become useful in certain contexts" only when they satisfy this law.

The Belinfante pair (4.4) is usually constructed on the basis of a tensor, antisymmetric in the first two indices, $\Delta\Upsilon^{\alpha\beta\kappa}=\Delta\Upsilon^{[\alpha\beta]\kappa}$:

$$\left(\Delta T^{\alpha\beta}\overset{\text{def}}{=}\partial_\kappa(\Delta\Upsilon^{\alpha\beta\kappa}-\Delta\Upsilon^{\beta\kappa\alpha}+\Delta\Upsilon^{\kappa\alpha\beta})/2,\ \Delta\Upsilon^{\alpha\beta\kappa}\right). \tag{4.4}$$

At its addition, there arises a set of pairs considered to be equivalent ones [1]:

$$\left(\underset{c}{T}{}^{\alpha\beta}+\partial_\kappa(\Delta\Upsilon^{\alpha\beta\kappa}-\Delta\Upsilon^{\beta\kappa\alpha}+\Delta\Upsilon^{\kappa\alpha\beta})/2,\ \underset{c}{\Upsilon}{}^{\alpha\beta\kappa}+\Delta\Upsilon^{\alpha\beta\kappa}\right). \tag{4.5}$$

From this set, the physicists choose a pair comprising, as it seems to them, a symmetric energy-momentum tensor and a zero spin tensor, which arise for

$$\Delta\Upsilon^{\alpha\beta\kappa}=-\underset{c}{\Upsilon}{}^{\alpha\beta\kappa},$$

$$\left(\underset{B}{T}{}^{\alpha\beta}=\underset{c}{T}{}^{\alpha\beta}-\partial_\kappa(\underset{c}{\Upsilon}{}^{\alpha\beta\kappa}-\underset{c}{\Upsilon}{}^{\beta\kappa\alpha}+\underset{c}{\Upsilon}{}^{\kappa\alpha\beta})/2,\ 0\right). \tag{4.6}$$

We name the tensor $\underset{B}{T}{}^{\alpha\beta}=\underset{c}{T}{}^{\alpha\beta}-\partial_\kappa(\underset{c}{\Upsilon}{}^{\alpha\beta\kappa}-\underset{c}{\Upsilon}{}^{\beta\kappa\alpha}+\underset{c}{\Upsilon}{}^{\kappa\alpha\beta})/2$ the Belinfante tensor.

The equivalence relation between the pairs of the set (4.5) is emphasized even by terminology. The transformation of the canonical pair (4.1) to the set (4.5) is named gauge, and it is said that the spin in the pair (4.6) is gauged to zero [1, p. 73]. It is proclaimed that the energy-momentum tensor in (4.6) is symmetric, and it gives a complete description of matter because the spin tensor $\underset{c}{\Upsilon}{}^{\alpha\beta\kappa}$ is entirely absorbed into $\underset{B}{T}{}^{\alpha\beta}$ during the symmetrization procedure. In other words, it is asserted [12] that "the momentum density gives rise to both the orbital angular momentum and the spin angular momentum" according to the simple formula

$$J^{ik}=L^{ik}+\Sigma^{ik}=\int 2x^{[i}\underset{s}{T}{}^{k]0}dV. \tag{4.7}$$

(We have to designate the spin by the symbol Σ because S is used below as torsion).

Certainly, this reasoning is erroneous from the beginning to the end even if we close our eyes at an inadmissible nature of any additions to the energy-momentum and spin tensors by virtue of such equalities as (1.1) and (2.2).

First, it is necessary to understand that the canonical pair (4.1) is obtained from the Lagrange formalism according to Noether's theorem. So, this pair satisfies the laws (4.2) only in the case of a free field, which does not interact with anything and, consequently, is unobservable and is not physical. An observation of the field momentum and spin implies its interaction with its sources. Divergences of the tensors serve as a criterion of their adequacy. The divergences determine the momentum and spin, which are exchanged by the field with its sources during an observation. Therefore, it is impossible to use the canonical tensors in an observation, because $\partial_\beta T^{\alpha\beta}$ must be zero, according to Nether's theorem.

Second, it is straightforward that if the equality (4.2) are ignored (in the presence of sources), the canonical tensors (4.1) contradict the reality. Indeed, e.g., the canonical Lagrangian of electromagnetism

$$\underset{c}{\Lambda}=-F_{\alpha\beta}F^{\alpha\beta}/4, \qquad F_{\alpha\beta}=2\partial_{[\alpha}A_{\beta]} \tag{4.8}$$

gives the canonical pair of tensors

$$\begin{aligned}\underset{c}{T}{}^{\alpha\beta} &= -\partial^\alpha A_\sigma F^{\beta\sigma}+g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}/4,\\ \underset{c}{\Upsilon}{}^{\alpha\beta\kappa} &= -2A^{[\alpha}F^{\beta]}{}_\kappa,\end{aligned} \tag{4.9}$$

which do not satisfy (4.2):

$$\partial_\beta \underset{c}{T}^{\alpha\beta} = -\partial^\alpha A_\sigma j^\sigma,$$
$$\partial_\kappa \underset{c}{\Upsilon}^{\alpha\beta\kappa} - 2\underset{c}{T}^{[\alpha\beta]} = -2A^{[\alpha}j^{\beta]}.$$

As is visible, the divergence of the canonical energy-momentum tensor is equal not to $-F^{\alpha\beta}j_\beta$, but to a wrong quantity. And this pair contradicts the experiment. For example, in a constant homogeneous magnetic field $B_x = B$, $B_y = B_z = 0$, for which

$$F_{yz} = F^{yz} = -B, \qquad A_y = zB/2, \qquad A_z = yB/2,$$

the stress tensor predicts an incorrect zero value of the field pressure across the lines of force:

$$\underset{c}{T}^{yy} = \underset{c}{T}^{zz} = 0.$$

Therefore, there is no sense to use a canonical pair for something practical.

Thirdly, addition of the Belinfante pair (4.4) to the canonical pair results, generally speaking, neither in a true tensor pair nor in a pair a with symmetric energy-momentum tensor. For example, in electrodynamics, addition of the Belinfante pair

$$\left(\Delta \underset{c}{T}^{\alpha\beta} = \partial_\sigma(A^\alpha F^{\beta\sigma}),\ -\underset{c}{\Upsilon}^{\alpha\beta\kappa}\right)$$
$$= (\partial_\sigma A^\alpha F^{\beta\sigma} - A^\alpha j^\beta,\ 2A^{[\alpha}F^{\beta]\kappa}) \qquad (4.10)$$

to the electromagnetic pair (4.9) gives an asymmetric energy-momentum tensor because of the term $-A^\alpha j^\beta$, though this addition eliminates the spin tensor. By the use the Belinfante procedure in electrodynamics, we obtain the pair

$$\left(\underset{B}{T}^{\alpha\beta} = -F^\alpha{}_\nu F^{\beta\nu} + g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}/4 - A^\alpha j^\beta,\ 0\right),$$

which contains, instead of the Maxwell tensor, the Belinfante tensor. This tensor has no physical sense. (It is designated $\hat{\Sigma}^{ij}[\tau]$ in Eq. (30) of [1].) Thus, the Belinfante procedure is useless, and the statements [9, 8] about an obligatory symmetry of an energy-momentum tensor are denied by the examples of Sec. 2 and 3.

These arguments remove the objections against the reality of the spin tensor in classical field theory. It is also necessary to note that even the real symmetry of a true energy-momentum tensor does not exclude the presence of a spin tensor. As was shown, the Maxwell tensor (1.3) is accompanied with the spin tensor (3.1). So, the angular momentum conservation law (2.3) may be violated for a particularly chosen medium. However, the absorption of spin results necessarily in an asymmetrical energy-momentum tensor for continuum theories.

Certainly, the standard formula (4.7) for the full angular momentum is not true and should be replaced with

$$J^{ik} = L^{ik} + S^{ik} = \int (2x^{[i}T^{k]0} + \Upsilon^{ik0})dV.$$

5. Metric energy-momentum tensor

Other objections against the reality of the spin tensor are related to the existence of an essentially symmetric metric energy-momentum tensor. As is known [13], an energy-momentum tensor of some field, named a metric tensor, may be obtained simply from the condition of invariance of the action

$$\mathcal{A} = \int \Lambda\sqrt{-g}d\Omega$$

under coordinate changes in Minkowski space because the field Lagrangian is a scalar. This tensor has the expression

$$\underset{g}{T}^{\alpha\beta} = \frac{2}{\sqrt{-g}}\left(\frac{\partial(\sqrt{-g}\Lambda}{\partial g_{\alpha\beta}} - \frac{\partial}{\partial x^\sigma}\frac{\partial(\sqrt{-g}\Lambda)}{\partial(\partial g_{\alpha\beta}/\partial x^\sigma)}\right), \quad (5.1)$$

and, in the case of the canonical Lagrangian of electromagnetism (4.8), appears as the Maxwell tensor (1.3). Besides that, an objection against the spin tensor is connected with the absence of a similar procedure in Minkowski space resulting in a spin tensor. Therefore, the spin tensor is considered not to exist.

In our opinion, these objections should be rejected even because the derivation of the tensor (5.1) cannot be recognized as a satisfactory one. The point is that, according to the derivation, the divergence of the tensor (5.1) is zero,

$$\nabla_\beta \underset{g}{T}^{\alpha\beta} = 0,$$

and it proves that the derivation is valid only for a free field in the absence of sources, as well as the derivation of the canonical pair (4.1). For, e.g., electrodynamics (4.8), it means that the tensor (5.1) may be used only under the condition $\nabla_\beta F^{\beta\sigma} = 0$, which excludes situations of interest. Therefore it is a happy circumstance that the tensor (5.1) coincides with the Maxwell tensor, which has a correct divergence and is true in the presence of currents. The canonical tensor in (4.9) is not so lucky.

The necessary symmetry of the metric tensor (5.1) proves its falseness since it contradicts the possible asymmetry of the energy-momentum tensor, shown in Sec. 3. So, the symmetry of the tensor (5.1) cannot serve as a reasoning against the spin tensor.

6. Equilibrium equations and equations of motion

Consider a piece of a rigid fibre of any form in the usual Euclidean space and two pairs of forces and a torque, $(\mathcal{F},\ \tau)$, which acts on the fibre at two points, so that the fiber is in equilibrium. If the point of application of one of the pairs is moving along the fibre, the pair must be changing to preserve the equilibrium:

$$d\mathcal{F}^i = 0, \qquad d\tau^{ij} = 2\mathcal{F}^{[i}dx^{j]}.$$

The change defines transport of a pair of a vector and a bivector, which has been called "fibre transport" [14, 15]. If curvilinear coordinates are used, the differentials and partial derivatives are replaced with covariant ones,

$$D\mathcal{F}^i = 0, \qquad D\tau^{ij} - 2\mathcal{F}^{[i}dx^{j]} = 0,$$

so there is obvious analogy between this transport and the equilibrium equations of matter in 3-space,

$$\nabla_k T^{ik} = 0, \qquad \nabla_k \Upsilon^{ijk} - 2T^{[ij]} = 0.$$

In the case of space-time, the fibre is replaced with a world line of a particle having a pair of momentum and spin (internal angular momentum) $(P^\alpha,\ \Sigma^{\alpha\beta})$. The transport of the pair satisfies the equations of motion of this particle,

$$DP^\alpha = 0, \quad D\Sigma^{\alpha\beta} - 2P^{[\alpha}dx^{\beta]} = 0, \tag{6.1}$$

which is analogous to the equilibrium equations of matter

$$\nabla_\beta T^{\alpha\beta} = 0, \qquad \nabla_\kappa \Upsilon^{\alpha\beta\kappa} - 2T^{[\alpha\beta]} = 0. \tag{6.2}$$

By the way, Eqs. (6.1) show that, in principle, the momentum can be transported in a direction different from its own direction, but the internal angular momentum of the carrier of this momentum is transported in a way different from its parallel transport.

As is known, in a transition to curved space (without torsion), an additional term appears in the equations of motion and in the equilibrium equations of matter. The term contains the curvature tensor, and the analogy between the equations of motion of a particle and the equilibrium equations of matter is kept [16, 17]:

$$\begin{aligned} DP^\alpha + R_{\mu\nu\kappa}^{\cdot\ \cdot\ \cdot\ \alpha}\Sigma^{\mu\nu}dx^\kappa/2 &= 0, \\ D\Sigma^{\alpha\beta} - 2P^{[\alpha}dx^{\beta]} &= 0, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \nabla_\kappa T^{\alpha\kappa} + R_{\mu\nu\kappa}^{\cdot\ \cdot\ \cdot\ \alpha}\Upsilon^{\mu\nu\kappa}/2 &= 0, \\ \nabla_\kappa \Upsilon^{\alpha\beta\kappa} - 2T^{[\alpha\beta]} &= 0. \end{aligned} \tag{6.4}$$

Eqs. (6.3) are Dixon's equations for a dipole particle. (The indices are arranged according to Schouten and Hehl).

Note that the Belinfante pair (4.4)

$$\left(T^{\alpha\beta} \overset{\text{def}}{=} \nabla_\kappa(\Upsilon^{\alpha\beta\kappa} - \Upsilon^{\beta\kappa\alpha} + \Upsilon^{\kappa\alpha\beta})/2,\ \Upsilon^{\alpha\beta\kappa}\right)$$

wonderfully satisfies the equilibrium equation (6.4) in curved space as well. Indeed,

$$\begin{aligned} \nabla_\beta T^{\alpha\beta} &\equiv \nabla_\beta\nabla_\kappa(\Upsilon^{\alpha\beta\kappa} - \Upsilon^{\beta\kappa\alpha} + \Upsilon^{\kappa\alpha\beta})/2 \\ &= \nabla_{[\beta\kappa]}(\Upsilon^{\alpha\beta\kappa} - \Upsilon^{\beta\kappa\alpha}/2) \\ &= [R_{\beta\kappa\mu}^{\cdot\ \cdot\ \cdot\ \alpha}(\Upsilon^{\mu\beta\kappa} - \Upsilon^{\beta\kappa\mu}/2) + R_{\kappa\mu}(\Upsilon^{\alpha\mu\kappa} \\ &\quad - \Upsilon^{\mu\kappa\alpha}/2) - R_{\beta\mu}(\Upsilon^{\alpha\beta\mu} - \Upsilon^{\beta\mu\alpha}/2)]/2. \end{aligned}$$

The terms with Ricci tensors equal zero, and, by the second identity $R_{[\beta\kappa\mu]}^{\cdot\ \cdot\ \cdot\ \alpha} = 0$, we have the first equality of (6.4):

$$\nabla_\beta T^{\alpha\beta} + R_{\mu\nu\kappa}^{\cdot\ \cdot\ \cdot\ \alpha}\Upsilon^{\mu\nu\kappa}/2 = 0,$$

while the second equation, $\nabla_\kappa \Upsilon^{\alpha\beta\kappa} - 2T^{[\alpha\beta]} = 0$, is obvious.

There is now a question, how Eqs. (6.3), (6.4) change in a transition to a space U_4 with torsion of the connection $S_{\beta\gamma}^{\cdot\ \cdot\ \alpha} = \Gamma^\alpha_{[\beta\gamma]}$? in this respect, we shall note that Eqs. (6.3), (6.4) do not depend on the metric tensor of space. They only depend on the connection coefficients $\Gamma^\nu_{\alpha\beta}$. This fact is natural since (6.3) represents transport of a vector and a bivector along a world line. The same principle, in our opinion, should be fair for a space with torsion. Therefore the standard equilibrium equations of matter in a space with torsion [17, (5)], [18, (3.12)],

$$\begin{aligned} &\overset{*}{\nabla}_\kappa\, T_\alpha^{\cdot\ \kappa} + S_{\kappa\alpha}^{\cdot\ \cdot\ \beta}T_\beta^{\cdot\ \kappa} - R_{\alpha\kappa}^{\cdot\ \cdot\ \mu\nu}\Upsilon_{\mu\nu}^{\cdot\ \cdot\ \kappa}/2 = 0, \\ &\overset{*}{\nabla}_\kappa\, \Upsilon^{\alpha\beta\kappa} - 2T^{[\alpha\beta]} = 0, \qquad \overset{*}{\nabla}_\kappa \overset{def}{=} \nabla_\kappa + 2S_{\kappa\beta}^{\cdot\ \cdot\ \beta}, \end{aligned} \tag{6.5}$$

may not be correct. Indeed, the definition of the curvature tensor

$$R_{\alpha\kappa\mu}^{\cdot\ \cdot\ \cdot\ \nu} = 2\partial_{[\alpha}\Gamma^\nu_{\kappa]\mu} + 2\Gamma^\nu_{[\alpha\beta}\Gamma^\beta_{\kappa]\mu}$$

requires the use of a metric tensor in (6.6):

$$\overset{*}{\nabla}_\kappa\, T_\alpha^{\cdot\ \kappa} + S_{\kappa\alpha}^{\cdot\ \cdot\ \beta}T_\beta^{\cdot\ \kappa} - R_{\alpha\kappa\mu}^{\cdot\ \cdot\ \cdot\ \nu}g_{\nu\rho}\Upsilon^{\mu\rho\kappa}/2 = 0,$$

and in the corresponding equation of motion [17]

$$DP_\alpha + S_{\kappa\alpha}^{\cdot\ \cdot\ \beta}P_\beta dx^\kappa - R_{\alpha\kappa\mu}^{\cdot\ \cdot\ \cdot\ \nu}g_{\nu\rho}\Sigma^{\mu\rho}dx^\kappa/2 = 0.$$

Besides, the transvection of the tensor, antisymmetric in $\mu\nu$, $\Upsilon_{\mu\nu}^{\cdot\ \cdot\ \kappa}$ and the tensor $R_{\alpha\kappa}^{\cdot\ \cdot\ \mu\nu}$ seems rather strange because $R_{\alpha\kappa}^{\cdot\ \cdot\ \mu\nu}$, is, generally speaking, not antisymmetric in the last indices.

7. Equilibrium equations in space U_4 (with torsion)

In this section, we shall offer a simple generalization of Eqs. (6.3), (6.4) for a space U_4. This generalization does not use a metric tensor. We postulate that the Belinfante pair (6.5) satisfies some equilibrium equations in a space U_4 with torsion, and we find the equations.

For brevity, we shall use the notation [19, p. 132],

$$\Upsilon^{\{\alpha\beta\kappa\}} = \Upsilon^{\alpha\beta\kappa} - \Upsilon^{\beta\kappa\alpha} + \Upsilon^{\kappa\alpha\beta}, \qquad \Upsilon^{\{\alpha\beta\kappa\}} = -\Upsilon^{\{\alpha\kappa\beta\}},$$

and remember that the energy-momentum and spin tensors are actually tensor densities of weight $+1$. This fact was ignored in the previous sections. Only in Sec. 5 the energy-momentum tensor density was written down in an explicit form, $T^{\alpha\beta}\sqrt{-g}$. Gothic symbols are usually applied to denote tensor densities. We shall, instead, mark the densities with the symbol 'wedge'. Such notation is due to I.A. Kunin's translation [21] of the book [20]. However, as distinct from [21], the wedge will be put at the level of bottom indices for a density of weight $+1$: $T^{\alpha\beta}_\wedge$, $\Upsilon^{\alpha\beta\kappa}_\wedge$, $\sqrt{-g}_\wedge$.

The covariant divergence of an antisymmetric contravariant density of weight $+\ 1$ is equal to its partial divergence [22, p. 387]. Therefore, in a space with torsion, keeping covariant forms, the covariant divergences ∇_κ, change into Cartan divergences, $\overset{c}{\nabla}_\kappa$. In our case we have

$$T^{\alpha\beta}_{\wedge} = \overset{c}{\nabla}_\kappa \Upsilon^{\{\alpha\beta\kappa\}}_{\wedge}/2 = (\partial_\kappa \Upsilon^{\{\alpha\beta\kappa\}}_{\wedge} + \Gamma^{\alpha}_{\kappa\gamma}\Upsilon^{\{\alpha\beta\kappa\}}_{\wedge})/2, \quad (7.1)$$

$$\overset{c}{\nabla}_\beta T^{\alpha\beta}_{\wedge} = \partial_\beta T^{\alpha\beta}_{\wedge} + \Gamma^{\alpha}_{\beta\gamma}T^{\alpha\beta}_{\wedge}. \quad (7.2)$$

Substitution of (7.1) to (7.2) yields

$$\begin{aligned}\overset{c}{\nabla}_\beta T^{\alpha\beta}_{\wedge} &= \partial_\beta(\partial_\kappa \Upsilon^{\{\alpha\beta\kappa\}}_{\wedge} + \Gamma^{\alpha}_{\kappa\gamma}\Upsilon^{\{\alpha\beta\kappa\}}_{\wedge})/2 \\ &+ \Gamma^{\alpha}_{\beta\gamma}(\partial_\kappa \Upsilon^{\{\gamma\beta\kappa\}}_{\wedge} + \Gamma^{\alpha}_{\kappa\delta}\Upsilon^{\{\delta\beta\kappa\}}_{\wedge})/2 \\ &= R_{\beta\kappa\gamma}{}^{\alpha}\Upsilon^{\{\gamma\beta\kappa\}}_{\wedge}/4 = R_{\{\kappa\gamma\beta\}}{}^{\alpha}\Upsilon^{\gamma\beta\kappa}_{\wedge}/4.\end{aligned}$$

As a result, instead of (6.6), we obtain the matter equilibrium equations in a space with torsion with the aid of the structure $R_{\{\kappa\gamma\beta\}}{}^{\alpha}$ and without a term connecting the torsion and momentum:

$$\begin{aligned}\overset{c}{\nabla}_\beta T^{\alpha\beta}_{\wedge} - R_{\{\kappa\gamma\beta\}}{}^{\alpha}/4 &= 0, \\ \overset{c}{\nabla}_\kappa \Upsilon^{\alpha\beta\kappa}_{\wedge} - 2T^{[\alpha\beta]}_{\wedge} &= 0. \end{aligned} \quad (7.3)$$

If $S_{\gamma\beta}{}^{\alpha} = 0$, then (7.3) turns into (6.4), since [22, p. 388]

$$\begin{aligned}R_{\{\kappa\gamma\beta\}}{}^{\alpha} &= -2R_{\gamma\beta\kappa}{}^{\alpha} + 3R_{[\kappa\gamma\beta]}{}^{\alpha} \\ &= -2R_{\gamma\beta\kappa}{}^{\alpha} + 6\,\overset{c}{\nabla}_{[\kappa}\, S_{\gamma\beta]}{}^{\alpha}.\end{aligned}$$

Note that another reasoning had earlier resulted in an additional term in (7.3), which connected the torsion and spin in the second equilibrium equation [23],

$$\begin{aligned}\overset{c}{\nabla}_\beta T^{\alpha\beta}_{\wedge} - R_{\{\kappa\gamma\beta\}}{}^{\alpha}/4 &= 0, \\ \overset{c}{\nabla}_\kappa \Upsilon^{\alpha\beta\kappa}_{\wedge} - 2T^{[\alpha\beta]}_{\wedge} - S_{\{\kappa\gamma}{}^{[\alpha}\delta^{\beta]}_{\rho\}}\Upsilon^{\gamma\rho\kappa}_{\wedge} &= 0. \end{aligned} \quad (7.4)$$

8. Conclusion

Thus, a substance may have classical spin as well as energy and momentum. This spin is described by a spin tensor just as the energy and momentum are described by the energy-momentum tensor. The true energy-momentum and spin tensors are unique and do not admit addition of any terms because they are observable quantities. These tensors localize the momentum and angular momentum. For electrodynamics, the true energy-momentum tensor is the Maxwell tensor, and the spin tensor was presented by the author in an article submitted to JETP earlier. Canonical tensors are not true. If a substance has no spin but absorbs it, the state of the substance is described by an asymmetric energy-momentum tensor because angular momentum conservation law connects the divergence of the spin tensor and the antisymmetric part of the energy-momentum tensor. The energy-momentum and spin conservation laws are referred to in the article as the matter equilibrium equations. These laws are similar to the equilibrium laws of an elastic fiber in three-dimensional space or to the equations of motion of a dipole particle in space-time. The equilibrium equations certainly, depends on the space curvature and torsion. The equilibrium equations in curved space without torsion are well known; they depend on the affine connection of space and do not depend explicitly on the metric tensor. However, for space with torsion, three variants of the equilibrium and motion equations are suggested. It is noted that one of them depends explicitly on the metric tensor, and it causes objection.

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