Beth's Experiment and Spin Tensor

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Abstract: It is shown that the result of the classical Beth's experiment with a circularly polarized beam and a half-wave plate cannot be explained if the electrodynamics spin is defined as a moment of a linear momentum. The fact is that the beam is used in the experiment along with its reflection from the mirror, so the resulting momentum density and moment of momentum density are zero everywhere. However, the result of the experiment is easy to explain if spin is defined by the spin tensor within the framework of the Lagrange formalism. According to this definition, the spin of a circularly polarized beam is not connected with the surface of the beam; spin is contained everywhere in electromagnetic radiation of circular polarization, including plane waves.

Key words: classical spin; circular polarization; electrodynamics
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1. Introduction.

Surprisingly, two mutually contradictory concepts of electrodynamics spin angular momentum $\mathbf{S}$ exist now. (Only waves with a flat phase front are considered here).

According to [1-8], the electrodynamics spin density is proportional to the *gradient* of the electromagnetic energy density. See details in [9]. Therefore, in particular, the spin of a circularly polarized light beam is the moment of the momentum density $\mathbf{E}\times\mathbf{H}/c^2$ or of the Poynting vector $\mathbf{E}\times\mathbf{H}$ that circulates on the surface of the beam. According to this concept, after averaging over time,

$$\mathbf{S} = <\int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV/c^2> = <\int \mathbf{r}\times d\mathbf{p}>. \tag{1.1}$$

According to another concept [10-11] (see also e.g. [12-22]) any circularly polarized light carries an angular momentum volume *density*, which is proportional to the electromagnetic energy density *itself*.

> **J.H. Poynting:** "If we put E for the energy in unit volume and G for the torque per unit area, we have $G = E\lambda/2\pi$" [11, p. 565].

Now this angular momentum density is described by a spin tensor $\Upsilon^{\lambda\mu\nu}$.

This concept does not associate spin with a linear momentum, or even with a motion of matter. **Hehl** writes about spin of an electron [23]:

> "The current density in Dirac's theory can be split into a convective part and a polarization part. The polarization part is determined by the spin distribution of the electron field. It should lead to *no* energy flux in the rest system of the electron because the genuine spin 'motion' take place only within a region of the order of the Compton wavelength of the electron".

So, in the framework of the Poynting conception, the beam's moment of momentum (1.1) is considered as an orbital angular momentum

$$\mathbf{L} = <\int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV/c^2> = <\int \mathbf{r}\times d\mathbf{p}>. \tag{1.2}$$

The electrodynamics spin tensor arises in the framework of the Lagrange formalism [24-26]. The sense of the spin tensor is as follows. The component Υ^{ij0} is a volume density of spin. This means that $dS^{ij} = \Upsilon^{ij0}dV$ is the spin of electromagnetic field inside the spatial element dV. The component Υ^{ijk} is a flux density of spin flowing in the direction of the x^k axis. For example, $dS_z/dt = dS^{xy}/dt = d\tau^{xy} = \Upsilon^{xyz}da_z$ is the z-component of spin passing through the surface element

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da_z per unit time, i.e. the torque acting on the element. Thus, the infinitesimal 4-volume dV_ν contains the spin angular momentum

$$dS^{\lambda\mu} = \Upsilon^{\lambda\mu\nu} dV_\nu\,, \text{ and } S^{\lambda\mu} = \int \Upsilon^{\lambda\mu\nu} dV_\nu\,. \tag{1.3}$$

We show that the classical Beth's experiment [27] confirms the Sadowsky and Poynting conception: electrodynamics angular momentum is proportional to the electromagnetic energy density. In the experiment, a circularly polarized beam passes through a half-wave plate, which changes the handedness to the opposite. So, the direction of the angular momentum of the beam changes, and the plate, according to the conservation principle, get double the angular momentum. However, to double the impact, the used beam passes through the plate a second time after reflection from the mirror.

Due to this circumstance, any circulation and even any movement of mass-energy is eliminated by adding the primary and reflected beams. A simple calculation shows that the Poynting vector is zero everywhere in the experiment (Section **2**). Thus, the quantity (1.1) called the spin angular momentum of the beam is zero in the experiment, and the experiment has no explanation in the framework of the concept [1-8].

At the sume time, the experiment is easy to explain if spin is defined by the spin tensor (1.3). The canonical spin tensor is used to explain the result of the Beth's experiment in Section **3**.

2. Poynting vector in the Beth's experment

A simple model of a wide circularly polarized electromagnetic beam with a plane phase front, directed along the axis *z*, is described by Jackson [8].

$$\mathbf{E}_1 = \exp(iz - it)[\mathbf{x} + i\mathbf{y} + \mathbf{z}(i\partial_x - \partial_y)]E_0(r)\,, \quad r^2 = x^2 + y^2\,, \tag{2.1}$$

$$\mathbf{H}_1 = \exp(iz - it)[-i\mathbf{x} + \mathbf{y} + \mathbf{z}(\partial_x + i\partial_y)]E_0(r)\,, \tag{2.2}$$

Here $\mathbf{E}_1$ и $\mathbf{H}_1$ are complex electromagnetic field vectors, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are unit coordinate vectors. ∂_x, ∂_y mean partial derivatives in *x* and *y*. For simplicity we put $\omega = k = c = \varepsilon_0 = \mu_0 = 1$. Index 1 means that formulas (2.1), (2.2) describe the primary beam after it passes through the half-wave plate. The amplitude of the beam is indicated $E_0(r)$. The function $E_0(r)$ is considered constant throughout the beam, i.e. when $r < R$, where R denotes the radius of the beam. However, on the surface of the beam, where $r \approx R$, the function quickly decreases to zero.

The reflected beam incident on the plate is marked by index 2. It has the same helicity as the primary beam (that is, it has the same *mutual* direction of the momentum and the spin). Therefore, the formulas for the reflected beam are obtained from formulas (2.1), (2.2) by changing the signs of *z* and *y*:

$$\mathbf{E}_2 = \exp(-iz - it)[\mathbf{x} - i\mathbf{y} + \mathbf{z}(-i\partial_x - \partial_y)]E_0(r)\,, \tag{2.3}$$

$$\mathbf{H}_2 = \exp(-iz - it)[-i\mathbf{x} - \mathbf{y} + \mathbf{z}(-\partial_x + i\partial_y)]E_0(r) \tag{2.4}$$

Adding the primary and reflected beam and writing out explicitly the real parts of the complex expressions, we get

$$E_x = \Re[\exp(iz - it) + \exp(-iz - it)]E_0 = 2E_0 \cos z \cos t\,, \tag{2.5}$$

$$E_y = \Re[i\exp(iz - it) - i\exp(-iz - it)]E_0 = -2E_0 \sin z \cos t\,, \tag{2.6}$$

$$\begin{aligned} E_z &= \Re[\exp(iz - it)(i\partial_x - \partial_y) + \exp(-iz - it)(-i\partial_x - \partial_y)]E_0 \\ &= -2(\sin z\partial_x + \cos z\partial_y)E_0 \cos t \end{aligned} \tag{2.7}$$

$$H_x = \Re[-i\exp(iz - it) - i\exp(-iz - it)]E_0 = -2E_0 \cos z \sin t\,, \tag{2.8}$$

$$H_y = \Re[\exp(iz - it) - \exp(-iz - it)]E_0 = 2E_0 \sin z \sin t\,, \tag{2.9}$$

$$\begin{aligned} H_z &= \Re[\exp(iz - it)(\partial_x + i\partial_y) + \exp(-iz - it)(-\partial_x + i\partial_y)]E_0 \\ &= 2(\sin z\partial_x + \cos z\partial_y)E_0 \sin t \end{aligned}\,, \tag{2.10}$$

and resulting electromagnetic field

$$\mathbf{E} = 2[\mathbf{x}\cos z - \mathbf{y}\sin z) - \mathbf{z}(\sin z\partial_x + \cos z\partial_y)]E_0 \cos t\,, \tag{2.11}$$

$$\mathbf{H} = -2[\mathbf{x}\cos z - \mathbf{y}\sin z) - \mathbf{z}(\sin z\partial_x + \cos z\partial_y)]E_0 \sin t\,. \tag{2.12}$$

It is seen that the electric and magnetic fields are parallel to each other everywhere. Therefore, the Poynting vector is zero.

3. Spin tensor in the Beth's experment

Now we calculate the spin flux in the resulting electromagnetic field (2.11), (2.12), which surrounds the plate using the component of the canonical spin tensor $\underset{c}{\Upsilon}^{\lambda\mu\nu}$ [24-26]

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}\,,\quad \underset{c}{\Upsilon}^{xyz} = -2A^{[x}F^{y]z} = A^x H_x + A^y H_y = \mathbf{A}\cdot\mathbf{H} \tag{3.1}$$

(here A^λ и $F_{\mu\nu}$ are the magnetic vector potential and the electromagnetic tensor). However, let us first consider this process qualitatively. Since the primary beam (2.1), (2.2) is right circularly polarized and is directed along the z axis, it creates a spin flux density in space $\underset{c}{\Upsilon}_1^{xyz} > 0$. The reflected beam incident on the plate also has right circular polarization. Therefore he carries the spin $S_z < 0$. But, moving against the z axis, it creates in space the same spin flux density as the primary beam, $\underset{c}{\Upsilon}_2^{xyz} > 0$. So the spins of the primary and reflected beams are summed, in contrast to the Poynting vectors, which are mutually eliminated.

To calculate the spin flux density using the formula (3.1), the vector potential is calculated previously.

$$\mathbf{A} = -\int \mathbf{E}dt = -2(\mathbf{x}\cos z - \mathbf{y}\sin z)\,E_0 \sin t\,, \tag{3.2}$$

$$\underset{c}{\Upsilon}^{xyz} = A^x H_x + A^y H_y = 4E_0^2 \sin^2 t,\quad <\underset{c}{\Upsilon}^{xyz}> = 2E_0^2\,, \tag{3.3}$$

A similar calculation on the other side of the plate gives the same result. Thus, the plate receives the resulting torque

$$\tau_{\mathrm{tot}} = 4\pi R^2 E_0^2 = 4P \tag{3.4}$$

according to the result of the Beth's experiment ($P = \pi R^2 E_0^2$ is the power of the beam).

4. Conclusion

Thus, the Beth's experiment confirms the Sadowsky & Poynting concept of electrodynamics angular momentum and the spin tensor adequacy.

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