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## Abstract

Jefimenko's generalization of the Coulomb and Bio-Savard laws is used to calculate the reaction of the radiation on the rotating electric dipole. It is found that the energy taken from the dipole is equal to the recognized value of the radiated energy. At the same time, it is confirmed that the angular momentum flux exceeds the generally accepted value by the spin radiation not seen before.

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# Radiation damping of a rotating dipole

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Jefimenko's generalization of the Coulomb and Bio-Savard laws is used to calculate the reaction of the radiation on the rotating electric dipole. It is found that the energy taken from the dipole is equal to the recognized value of the radiated energy. At the same time, it is confirmed that the angular momentum flux exceeds the generally accepted value by the spin radiation not seen before.

## 1. Introduction

As shown in articles [1–4], the radiation of a rotating electric dipole contains, in addition to the generally recognized angular momentum flux [1–9]

$$dL_z/dt = \omega^3 p^2/6\pi\varepsilon_0 c^3, \tag{1.1}$$

which is localized mainly near the equatorial plane, with the angular distribution

$$dL_z/dtd\Omega = \omega^3 p^2 \sin^2\theta/16\pi^2\varepsilon_0 c^3, \tag{1.2}$$

also the spin flux

$$dS_z/dt = \omega^3 p^2/12\pi\varepsilon_0 c^3, \tag{1.3}$$

which is localized mainly near the axis of rotation of the dipole, with the angular distribution

$$dS_z/dtd\Omega = \omega^3 p^2 \cos^2\theta/16\pi^2\varepsilon_0 c^3 \tag{1.4}$$

($d\Omega = \sin\theta d\theta d\phi$, and the unit system is used in which $\mathrm{div}\mathbf{E} = \rho/\varepsilon_0$).
Thus, the total angular momentum flux is

$$dJ_z/dt = dL_z/dt + dS_z/dt = \omega^3 p^2/4\pi\varepsilon_0 c^3. \tag{1.5}$$

Meanwhile, the power radiated by a rotating dipole is recognized as

$$P = \omega^4 p^2/6\pi\varepsilon_0 c^3, \tag{1.6}$$

and, therefore, the usual mechanics equality

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$$P = \omega dJ/dt. \tag{1.7}$$

is violated

The result (1.5) was obtained in [2–4] by integrating the sum of the spin tensor and the moment of the Poynting vector over the surface surrounding a rotating dipole. In [1], this result was confirmed using the quantum mechanical Feynman proof and, independently, calculating the reaction of the magnetic vector potential field on the dipole.

In the present work, the energy (1.6) and angular momentum (1.5) fluxes are calculated using the same type by the use of the retarded electromagnetic Jefimenko's field [10]. It turned out that the forces acting on the dipole and responsible for the loss of energy differ from the forces responsible for the loss of angular momentum.

As a rotating electric dipole, we consider a pair of oscillating dipoles perpendicular to each other and having a quarter-period oscillation shift in time,

$$p^x = p\exp(-i\omega t), \quad p^y = ip\exp(-i\omega t). \tag{1.8}$$

## 2. Energy loss by a rotating dipole

The value (1.6) is calculated as the result of the influence of the electromagnetic field of the dipole on the dipole itself, according to the formula for the density of the resulting power

$$P_\wedge = -(\mathbf{j}\cdot\mathbf{E}); \tag{2.1}$$

here $\mathbf{j}$ and $\mathbf{E}$ are current density flowing along the dipole and the electric field in the dipole, respectively, and the index $\wedge$ for $P_\wedge$ means, in this case, "volume density", $dP = P_\wedge d^3x$. First, the effect of the $x$-dipole on itself is calculated.

In this paper, the electric field near the dipole is calculated from the known formula, taking into account the retardation [10 (6.55)]:

$$\mathbf{E}(\mathbf{x}, t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{\frac{\hat{\mathbf{r}}}{r^2}[\rho(\mathbf{x}', t')]_{\text{ret}} + \frac{\hat{\mathbf{r}}}{cr}[\partial_{t'}\rho(\mathbf{x}', t')]_{\text{ret}} - \frac{1}{c^2 r}[\partial_{t'}\mathbf{j}(\mathbf{x}', t')]_{\text{ret}}\right\}, \tag{2.2}$$

and an "elementary vibrator" is considered as a dipole; the current of the dipole is the same at all points, and the charges are only at the ends (see Fig. 1)

The dipole current $I_x$ is obtained by differentiating the charge

$$p^x = ql\exp(-i\omega t), \tag{2.3}$$

$$I_x = \partial_t p^x/l = -i\omega q\exp(-i\omega t). \tag{2.4}$$

The first term of expression (2.2),

$$E_1^x = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{\frac{1}{r^2}[\rho(\mathbf{x}', t')]_{\text{ret}}\right\},$$

is simply the retarded Coulomb field at the point $x$. Therefore, replacing

$$d^3x'\rho \to d\widetilde{q}', \quad t \to t - (l/2 \pm x)/c, \quad r \to l/2 \pm x$$

and taking into account the direction of the electric field, we obtain the electric field strength from both charges at the point $x$:

$$E_1^x(x) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-\exp[-i\omega t + i\omega(l/2 + x)/c]}{(l/2 + x)^2} - \frac{\exp[-i\omega t + i\omega(l/2 - x)/c]}{(l/2 - x)^2}\right]. \tag{2.5}$$

The corresponding contribution of this term to the power generated by the dipole is given by formula (2.1) (we replace $jd^3x \to Idx$ from (5), and the bar means complex conjugation)

$$\begin{aligned} P_1 &= \frac{-1}{2}\int_{-l/2}^{l/2} dx\Re\{\bar{I}E_1^x\} \\ &= -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx\Re\left\{i\exp(i\omega t)\left[\frac{-\exp[-i\omega t + i\omega(l/2 + x)/c]}{(l/2 + x)^2} - \frac{\exp[-i\omega t + i\omega(l/2 - x)/c]}{(l/2 - x)^2}\right]\right\} \\ &= -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx\left[\frac{\sin[\omega(l/2 + x)/c]}{(l/2 + x)^2} + \frac{\sin[\omega(l/2 - x)/c]}{(l/2 - x)^2}\right]. \end{aligned} \tag{2.6}$$

Taking into account the small size of the dipole, we consider only two terms of the expansion of the sine in a series

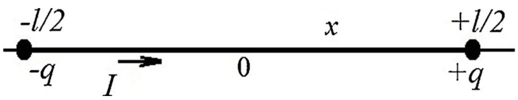


**Fig. 1.** $x$-dipole.

$$P_1 = -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx\left[\frac{\omega}{c(l/2+x)} - \frac{\omega^3(l/2+x)}{6c^3} + \frac{\omega}{c(l/2-x)} - \frac{\omega^3(l/2-x)}{6c^3}\right]P_1$$
$$= -\frac{\omega q^2}{8\pi\varepsilon_0}\int_{-l/2}^{l/2} dx\left[\frac{\omega}{c(l/2+x)} - \frac{\omega^3(l/2+x)}{6c^3} + \frac{\omega}{c(l/2-x)} - \frac{\omega^3(l/2-x)}{6c^3}\right]. \tag{2.7}$$

Similarly to formula (2.5), we find the electric field provided by the second term of formula (2.2)

$$E_2^x = \frac{i\omega q}{4\pi\varepsilon_0 c}\left[\frac{\exp[-i\omega t + i\omega(l/2+x)/c]}{(l/2+x)} + \frac{\exp[-i\omega t + i\omega(l/2-x)/c]}{(l/2-x)}\right]. \tag{2.8}$$

In contrast to formula (2.5), this formula contains *i.*

Formula (2.1) gives the contribution of this term, $E_2^x$, to the power generated by the dipole

$$P_2 = \frac{\omega^2 q^2}{8\pi\varepsilon_0 c}\int_{-l/2}^{l/2} dx\Re\left\{\exp(i\omega t)\left[\frac{\exp[-i\omega t + i\omega(l/2+x)/c]}{(l/2+x)} + \frac{\exp[-i\omega t + i\omega(l/2-x)/c]}{(l/2-x)}\right]\right\}$$
$$= \frac{\omega^2 q^2}{8\pi\varepsilon_0 c}\int_{-l/2}^{l/2} dx\left[\frac{\cos[\omega(l/2+x)/c]}{(l/2+x)} + \frac{\cos[\omega(l/2-x)/c]}{(l/2-x)}\right]. \tag{2.9}$$

Restricting ourselves to two terms of the expansion of cosine in a series, we have

$$P_2 = \frac{\omega^2 q^2}{8\pi\varepsilon_0 c}\int_{-l/2}^{l/2} dx\left[\frac{1}{(l/2+x)} - \frac{\omega^2(l/2+x)}{2c^2} + \frac{1}{(l/2-x)} - \frac{\omega^2(l/2-x)}{2c^2}\right]. \tag{2.10}$$

Surprisingly, the integrals diverging at the ends of the dipole are shortened upon the addition $P_1 + P_2$, and the remaining terms are constants. As $l/6 - l/2 = -l/3$, this part of the power is

$$P_1 + P_2 = -\frac{\omega^4 p^2}{24\pi\varepsilon_0 c^3}. \tag{2.11}$$

The third term of formula (2.2),

$$E_3^x = \frac{1}{4\pi\varepsilon_0}\int dx'\left\{-\frac{1}{c^2 r}[\partial_{t'} I(x', t']_{\text{ret}}\right\},$$

uses the derivative of the current

$$\partial_t I_x = -\omega^2 q \exp(-i\omega t) \tag{2.12}$$

To calculate the strength at the point $x$, we divided the region of integration into two parts by the point $x$

$$E_3^x = -\frac{\omega^2 q}{4\pi\varepsilon_0 c^2}\left\{\int_{-l/2}^{x} dx'\frac{-\exp[-i\omega t + i\omega(x-x')/c}{x-x'} + \int_{x}^{l/2} dx'\frac{-\exp[-i\omega t + i\omega(x'-x)/c}{x'-x}\right\}. \tag{2.13}$$

Using formula (2.1), $dP = -(\mathbf{j}\cdot\mathbf{E})d^3x = -IE^x dx$, and current (2.4), we obtain the power corresponding to the third term of formula (2.2)

$$P_3 = -\frac{\omega^3 q^2}{8\pi\varepsilon_0 c^2}\int_{-l/2}^{l/2} dx\Re\left\{i\omega\exp(i\omega t)\left[\int_{-l/2}^{x} dx'\frac{\exp[-i\omega t + i\omega(x-x')/c]}{x-x'} + \int_{x}^{l/2} dx'\frac{\exp[-i\omega t + i\omega(x'-x)/c]}{x'-x}\right]\right\}$$
$$= -\frac{\omega^3 q^2}{8\pi\varepsilon_0 c^2}\int_{-l/2}^{l/2} dx\left[\int_{-l/2}^{x} dx'\frac{-\sin[\omega(x-x')/c]}{x-x'} + \int_{x}^{l/2} dx'\frac{-\sin[\omega(x'-x)/c]}{x'-x}\right]. \tag{2.14}$$

Restricting ourselves to one term in the expansion of the sine in a series, we easily obtain

$$P_3 = \frac{\omega^4 \mathrm{d}^2}{8\pi\varepsilon_0 c^3}. \tag{2.15}$$

Thus, the power radiated by one *x*-dipole is

$$P = P_1 + P_2 + P_3 = \omega^4 p^2/(12\pi\varepsilon_0 c^3). \tag{2.16}$$

Naturally, the y-dipole, acting on itself, makes the same contribution. It will be shown below that the electric field of one oscillating dipole in the territory of another oscillating dipole is perpendicular to the current, and therefore does not produce energy. So a *rotating* dipole delivers power (1.6) to the radiation:

$$P = \omega^4 p^2/6\pi\varepsilon_0 c^3$$

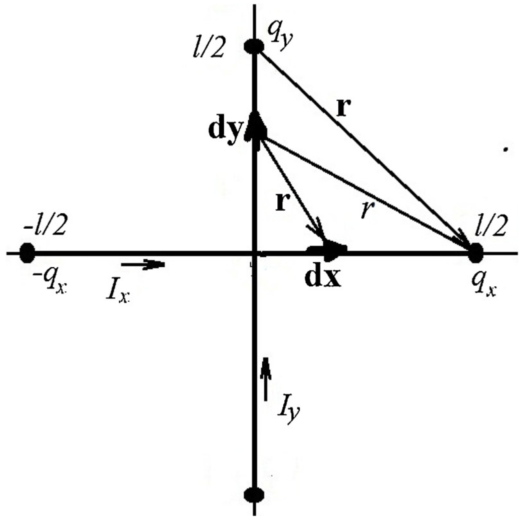


**Fig. 2.** The pair of oscillating dipoles. Current elements and radii used in the formulas are indicated.

## 3. The torque experienced by the charges of a rotating dipole

We now calculate the electric field **E** created by the $y$-dipole at the location of the charge of the $x$-dipole, that is, at the point $x = l/2$ (see Fig. 2)

To shorten the notation, we rewrite formula (2.2) in terms of charges and currents:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{\hat{\mathbf{r}}}{r^2}[q_y]_{\text{ret}} + \frac{\hat{\mathbf{r}}}{cr}\left[\partial_{t'}q_y\right]_{\text{ret}} - \int_{-l/2}^{l/2}\frac{1}{c^2r}\left[\partial_{t'}I_y\right]_{\text{ret}}\mathbf{dy}\right); \tag{3.1}$$

here the charge and current belong to the $y$-dipole:

$$q_y = iq\exp(-i\omega t),\ \ [q_y]_{\text{ret}} = iq\exp(-i\omega t + i\omega r/c),\ \ \left[\partial_{t'}q_y\right]_{\text{ret}} = \omega q\exp(-i\omega t + i\omega r/c) \tag{3.2}$$

$$I_y = \partial_t q_y = \omega q\exp(-i\omega t),\ \ [I_y]_{\text{ret}} = \omega q\exp(-i\omega t + i\omega r/c).\ \ \left[\partial_{t'}I_y\right]_{\text{ret}} = -i\omega^2 q\exp(-i\omega t + i\omega r/c) \tag{3.3}$$

The electric field created by the charge consists of two terms:

$$\mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{r}}([q_y]_{\text{ret}}/r^2 + \left[\partial_{t'}q_y\right]_{\text{ret}}/cr)$$
$$/4\pi\varepsilon_0\, I_y = \partial_t q_y = \omega q\exp(-i\omega t),\ \ [I_y]_{\text{ret}} = \omega q\exp(-i\omega t + i\omega r/c).\ \ \left[\partial_{t'}I_y\right]_{\text{ret}} = -i\omega^2 q\exp(-i\omega t + i\omega r/c). \tag{3.4}$$

However, the forces created by these terms on the charge $q_x = q\exp(-i\omega t)$ are mutually eliminated when the dipole size tends to zero, although they tend to infinity.

$$\begin{aligned}F_1 + F_2 &= \Re\{(E_1 + E_2)\,\bar{q}_x\}/2 = \Re\{i\exp(i\omega r/c)/r^2 + \omega\exp(i\omega r/c)/cr\}q^2/8\pi\varepsilon_0\\ &= \{-\sin(\omega r/c)/r^2 + \omega\cos(\omega r/c)/cr\}q^2/8\pi\varepsilon_0 \to (-\omega/cr + \omega/cr)q^2/8\pi\varepsilon_0 = 0.\end{aligned} \tag{3.5}$$

Similarly, the total interaction forces of other pairs of charges are zero. So the damping of the rotating dipole is provided only by the third term of the formula (3.1):

$$\mathbf{E}_3 = -\int_{-l/2}^{l/2}\frac{1}{c^2r}[-i\omega^2 q\exp(-i\omega t + i\omega r/c)]\mathbf{dy} \tag{3.6}$$

The force acting on the charge $q_x$ along the $y$-axis is

$$\begin{aligned}F_3 &= \Re\{E_3\bar{q}_x\}/2 = \Re\{\int_{-l/2}^{l/2}\frac{1}{c^2r}i\omega^2q^2\exp(i\omega r/c)dy\}/8\pi\varepsilon_0.\\ &= -\int_{-l/2}^{l/2}\frac{1}{c^2r}\omega^2q^2\sin(\omega r/c)dy\}/8\pi\varepsilon_0 \to -\int_{-l/2}^{l/2}\omega^3q^2dy/8\pi\varepsilon_0c^3 = -\omega^3q^2l/8\pi\varepsilon_0c^3.\end{aligned} \tag{3.7}$$

The torque acting on both charges of the $x$-dipole is $-\ \omega^3q^2l^2/8\pi\varepsilon_0c^3$. Therefore, the torque acting on the rotating dipole is directed against the rotation of the dipole and is equal to (1.5)

$\omega^3 p^2/4\pi\varepsilon_0 c^3$.

## 4. Magnetic field torque

In addition to the electric field (2.2), Jefimenko's formulas give the magnetic field [10 (6.56)],

$$\mathbf{B}(\mathbf{x}, t) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\left\{[\mathbf{j}(\mathbf{x}', t')]_{\rm ret}\times\frac{\hat{\mathbf{r}}}{r^2} + \left[\partial_{t'}\mathbf{j}\left(\mathbf{x}', t'\right)\right]_{\rm ret}\times\frac{\hat{\mathbf{r}}}{cr}\right\}, \tag{4.1}$$

This field acts on dipoles (1.8) by the Lorentz force. Consider the magnetic field created by the $y$-dipole on the territory of the x-dipole. By analogy with (3.1), we write

$$\mathbf{B} = \frac{1}{4\pi\varepsilon_0}\int([I_y]_{\rm ret}/r^2 + \left[\partial_{t'}I_y\right]_{\rm ret}/cr)\mathrm{d}\mathbf{y}\times\hat{\mathbf{r}} \tag{4.2}$$

However, using (3.3) and (2.4), we find that the average value of the Lorentz force acting on the $x$-dipole is zero:

$$\begin{aligned}\mathbf{F} &= \Re\{\int \bar{I}_x\mathrm{d}\mathbf{x}\times\mathbf{B}\}/2 = \Re\{\iint i\omega q[\omega q\exp(i\omega r/c)/r^2 - i\omega^2 q\exp(i\omega r/c)/cr]\mathrm{d}\mathbf{x}\times(\mathrm{d}\mathbf{y}\times\hat{\mathbf{r}})\}/8\pi\varepsilon_0\\ &= \iint[-\sin(\omega r/c)\omega^2/r^2 + \cos(\omega r/c)\omega^3/cr]q^2\mathrm{d}\mathbf{x}\times(\mathrm{d}\mathbf{y}\times\hat{\mathbf{r}})/8\pi\varepsilon_0 \to 0 \quad \text{if} \quad r\to 0.\end{aligned} \tag{4.3}$$

So the total torque acting on the rotating dipole is (1.5)

## 5. Conclusion

This calculation confirms the presence of spin radiation by a rotating dipole, the radiation that was predicted using the spin tensor. This proves that the spin of electromagnetic radiation is actually described by the spin tensor. The spin tensor has been successfully used to describe also plane waves and beams [11–17].

I am eternally grateful to Professor Robert Romer for the courageous publication of my question: "Does a plane wave really not carry spin?" (was submitted on 07 October 1999) [18].

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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