Inevitability of the electrodynamics' spin tensor

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It is shown that the standard Lagrange formalism does not give the Maxwell energy-momentum tensor of electrodynamics and, to make matter worse, gives the false impression that an electrodynamics' spin tensor equals zero! A modified use of the canonical energy-momentum and spin tensors has led to an electrodynamics' spin tensor. A series of theoretical and experimental works confirms reality of the spin tensor and proves, in particular, that a circularly polarized light beam with plane phase front carries an angular momentum flux, which equals two power of the beam divided by the frequency. This fact contradicts the standard electrodynamics, which predicts the beam's angular momentum flux equals power of the beam divided by frequency, and means the electrodynamics is incomplete.

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1. Does electrodynamics' spin tensor exist?

As is well known, photons, i.e. electromagnetic waves, carry spin, energy, momentum, and angular momentum that is a moment of the momentum relative to a given point or to a given axis. Energy and momentum of electromagnetic waves are described by the Maxwell energy-momentum tensor (density)

$$T^{\lambda\mu} = -g^{\lambda\alpha}F_{\alpha\nu}F^{\mu\nu} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4\,, \tag{1.1}$$

where $F^{\mu\nu} = -F^{\nu\mu}$, $F_{\mu\nu} = F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ is the field strength tensor. For example, T^{i0} is a volume density of the momentum (quantity of moion) of electromagnetic waves, i.e., $dP^i = T^{i0}dV = \mathbf{E}\times\mathbf{B}\,dV$ is the momentum of waves inside of the infinitesimal volume dV, and $P^i = \int_V T^{i0}dV = \int_V \mathbf{E}\times\mathbf{B}dV$ is the momentum of waves inside of the arbitrary volume *V*. T^{0i} is a flux density of energy, i.e., $dW = T^{0i}da_i dt = (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}dt$ is the energy that has flowed through the infinitesimal area da_i in the time dt, and $dW = \int_a T^{0i}da_i dt = \int_a (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}dt$ is the energy that has flowed through the arbitrary area *a* in the time dt. (We duplicate the tensor notations by the vector notations when it is possible). We set $c = \varepsilon_0 = \mu_0 = 1$.

An interaction between electromagnetic waves and substance is described by a divergence of the energy-momentum tensor $\partial_\mu T^{\lambda\mu}$, i.e. by the Lorentz force density, viz.,

$$f^\lambda = -\partial_\mu T^{\lambda\mu} = F^{\lambda\beta}\partial^\mu F_{\mu\beta} = j_\beta F^{\lambda\beta}\,. \tag{1.2}$$

The Maxwell equations $\partial_{[\lambda}F_{\mu\nu]} = 0$, $\partial^\mu F_{\mu\beta} = j_\beta$ .are used here.

The angular momentum that is a moment of the momentum can be defined as [1]

$$L^{ij} = \int_V 2x^{[i}T^{j]0}dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV\,, \tag{1.3}$$

and this construction must be named as an orbital angular momentum. However, the modern electrodynamics has no describing of spin, though a concept of classical spin, which differs from the moment of momentum, is contained in the modern theory of fields. Unfortunately, the concept of spin is smothered in the standard electrodynamics as will be shown below.

Realy, the electrodynamics starts from the canonical Lagrangian [2 (4-111)], $\underset{c}{\mathsf{L}} = -F_{\mu\nu}F^{\mu\nu}/4$.

Then, by the Lagrange formalism, the canonical energy-momentum tensor [2 (4-113)]

$$\underset{c}{T}{}^{\lambda\mu} = \partial^\lambda A_\alpha \frac{\partial \underset{c}{\mathsf{L}}}{\partial(\partial_\mu A_\alpha)} - g^{\lambda\mu}\underset{c}{\mathsf{L}} = -\partial^\lambda A_\alpha F^{\mu\alpha} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4 \tag{1.4}$$

and the canonical total angular momentum tensor [2 (4-147)]

$$\underset{c}{J}^{\lambda\mu\nu} = 2x^{[\lambda}\underset{c}{T}^{\mu]\nu} + \underset{c}{\Upsilon}^{\lambda\mu\nu} \tag{1.5}$$

are obtained. Here

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial \underset{c}{L}}{\partial(\partial_\nu A_\alpha)} = -2A^{[\lambda}F^{\mu]\nu}, \tag{1.6}$$

is the canonical spin tensor [2 (4-150)]. Its space component is $\mathbf{E}\times\mathbf{A}$:

$$\underset{c}{\Upsilon}^{ij0} = \mathbf{E}\times\mathbf{A}, \tag{1.7}$$

The sense of a spin tensor $\Upsilon^{\lambda\mu\nu}$ is as follows. The component Υ^{ij0} is a volume density of spin. This means that $dS^{ij} = \Upsilon^{ij0}dV$ is the spin of electromagnetic field inside the spatial element dV. The component Υ^{ijk} is a flux density of spin flowing in the direction of the x^k axis. For example, $dS_z/dt = dS^{xy}/dt = d\tau^{xy} = \Upsilon^{xyz}da_z$ is the *z*-component of spin flux passing through the surface element da_z per unit time, i.e. the torque acting on the element.

The sense of a total angular momentum tensor, $J^{\lambda\mu\nu}$, is that the total angular momentum in an element dV_ν is $dJ^{\lambda\mu} = J^{\lambda\mu\nu}dV_\nu = 2x^{[\lambda}T^{\mu]\nu}dV_\nu + \Upsilon^{\lambda\mu\nu}dV_\nu$. The corresponding integral is

$$J^{\lambda\mu} = L^{\lambda\mu} + S^{\lambda\mu} = \int_V 2x^{[\lambda}T^{\mu]\nu}dV_\nu + \int_V \Upsilon^{\lambda\mu\nu}dV_\nu . \tag{1.8}$$

It consists of two terms: the first term involves a moment of momentum and represents an orbital angular momentum; the second term is spin. It must be emphasized that a moment of momentum cannot represent spin. This idea is discussed in the paper [3], which was written in response to [4]

However, the canonical tensors (1.4), (1.5), (1.6) are not electrodynamics tensors. They obviously contradict experiments. For example, consider a uniform electric field:

$$A_0 = -Ex,\ \ A_x = 0,\ \ \partial_\alpha A^\alpha = 0,\ \ F_{x0} = -F^{x0} = \partial_x A_0 = -E . \tag{1.9}$$

The canonical energy density (1.4) is negative:

$$\underset{c}{T}^{00} = g^{00}F_{x0}F^{x0}/2 = -E^2/2 . \tag{1.10}$$

Another example: consider a circularly polarized plane wave (or a central part of a corresponding light beam),

$$E^x = \cos(z-t),\ E^y = -\sin(z-t),\ B^x = \sin(z-t),\ B^y = \cos(z-t),\ \ A^x = \sin(z-t),\ A^y = \cos(z-t) \tag{1.11}$$

(for short we set $k = \omega = 1$). A calculation of components of the canonical spin tensor (1.6) yields

$$\underset{c}{\Upsilon}^{xy0} = 1,\quad \underset{c}{\Upsilon}^{xyz} = 1,\quad \underset{c}{\Upsilon}^{zxy} = A^x B_x = \sin^2(z-t),\quad \underset{c}{\Upsilon}^{yzx} = A^y B_y = \cos^2(z-t). \tag{1.12}$$

This result is absurd because, though $\underset{c}{\Upsilon}^{xy0}$ and $\underset{c}{\Upsilon}^{xyz}$ are adequate, the result means that there are spin fluxes in y & x - directions, i.e. in the directions, which are transverse to the direction of the wave propagation.

An opinion exists that a change of the Lagrangian can help to obtain the Maxwell tensor (1.1). A. Barut [5] presented a series of Lagrangians and field equations in Table 1

Table I

Lagrangians and Equations of Motion for the Most Common Fields

Field	Lagrangian	Field Equations
Free Electromagnetic Field	$L_{\mathrm{I}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2 - B^2)$	$F^{\mu\nu}{}_{,\nu} = 0$
	$L_{\mathrm{II}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(A^\mu{}_{,\mu})^2$	$\Box^2 A_\mu = 0$
	$L_{\mathrm{III}} = -\frac{1}{2}A^\mu{}_{,\nu}A_\mu{}^{,\nu}$	$\Box^2 A_\mu = 0$
	$L_{\mathrm{IV}} = \frac{1}{2}[A_\nu F^{\mu\nu}{}_{,\mu} - A_{\nu,\mu}F^{\mu\nu}] + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$	$\Box^2 A_\mu = 0$
Electromagnetic Field with an External Current	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\mu j^\mu$	$F^{\mu\nu}{}_{,\nu} = -\frac{1}{c}j^\mu$

However, A. Barut did not show energy-momentum and spin tensors corresponding to these Lagrangians. So, we add Table 2

Table 2

Electrodynamics' Lagrangians, Energy-Momentum Tensors, and Spin Tensors

Lagrangian	Energy-momentum tensor	Spin tensor
$L_I = \underset{c}{L} = -F_{\mu\nu}F^{\mu\nu}/4$	$T_I^{\lambda\mu} = \underset{c}{T}{}^{\lambda\mu} = -A_\nu{}^{,\lambda}F^{\mu\nu} + g^{\lambda\mu}F_{\sigma\nu}F^{\sigma\nu}/4$	$\Upsilon_I^{\lambda\mu\nu} = \underset{c}{\Upsilon}{}^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}$
$L_{II} = -F_{\mu\nu}F^{\mu\nu}/4 - (A^\mu{}_{,\mu})^2/2$	$T_{II}^{\lambda\mu} = T_I^{\lambda\mu} - A^{\mu,\lambda}A^\sigma{}_{,\sigma} + g^{\lambda\mu}(A^\sigma{}_{,\sigma})^2/2$	$\Upsilon_{II}^{\lambda\mu\nu} = \Upsilon_I^{\lambda\mu\nu} + 2A^{[\lambda}g^{\mu]\nu}A^\sigma{}_{,\sigma}$
$L_{III} = -A^\mu{}_{,\nu}A_\mu{}^{,\nu}/2$	$T_{III}^{\lambda\mu} = -A_\sigma{}^{,\lambda}A^{\sigma,\mu} + g^{\lambda\mu}A_{\sigma,\rho}A^{\sigma,\rho}$	$\Upsilon_{III}^{\lambda\mu\nu} = 2A^{[\lambda}A^{\mu],\nu}$
$L_V = -F_{\mu\nu}F^{\mu\nu}/4 - A_\sigma j^\sigma$	$T_V^{\lambda\mu} = T_I^{\lambda\mu} + g^{\lambda\mu}A_\sigma j^\sigma$	$\Upsilon_V^{\lambda\mu\nu} = \Upsilon_I^{\lambda\mu\nu}$

It is clear, none of these energy-momentum tensors is the Maxwell tensor. And what is more, none of these tensors differs from the Maxwell tensor by a divergence of an antisymmetric quantity. In other words, none of these tensors has true divergence (1.2). A method is unknown to get a tensor with the true divergence in the frame of the standard Lagrange formalism. A desire for such a tensor led Professor Soper to a mistake [6]. He used Lagrangian L_V, but, instead of the tensor $T_V^{\lambda\mu}$, he arrived at a false tensor [6, (8.3.5) – (8.3.9)]

$$\underset{f}{T}{}^{\lambda\mu} = T_I^{\lambda\mu} + A^\lambda j^\mu, \tag{1.13}$$

which differs from the Maxwell tensor by a divergence of an antisymmetric quantity:

$$T^{\lambda\mu} - \underset{f}{T}{}^{\lambda\mu} = \partial_\alpha A^\lambda F^{\mu\alpha} - A^\lambda j^\mu = \partial_\alpha (A^\lambda F^{\mu\alpha}). \tag{1.14}$$

In the frame of the standard procedure, a specific terms,

$$\underset{st}{t}{}^{\lambda\mu} = -\partial_\nu \tilde{\Upsilon}^{\lambda\mu\nu}/2 \tag{1.15}$$

and

$$\underset{st}{m}{}^{\lambda\mu\nu} = -\partial_\kappa (x^{[\lambda}\tilde{\Upsilon}^{\mu]\nu\kappa}), \tag{1.16}$$

are added to the canonical tensors (1.4) and (1.5) [7, 8] (here $\tilde{\Upsilon}^{\lambda\mu\nu} \overset{def}{=} \underset{c}{\Upsilon}{}^{\lambda\mu\nu} - \underset{c}{\Upsilon}{}^{\mu\nu\lambda} + \underset{c}{\Upsilon}{}^{\nu\lambda\mu} = -2A^\lambda F^{\mu\nu}$). This procedure gives a standard energy-momentum tensor $\underset{st}{T}{}^{\lambda\mu}$ and a standard total angular momentum tensor $\underset{st}{J}{}^{\lambda\mu\nu}$,

$$\underset{st}{T}{}^{\lambda\mu} = \underset{c}{T}{}^{\lambda\mu} + \underset{st}{t}{}^{\lambda\mu} = -\partial^\lambda A_\nu F^{\mu\nu} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4 + \partial_\nu (A^\lambda F^{\mu\nu}), \tag{1.17}$$

$$\underset{st}{J}{}^{\lambda\mu\nu} = \underset{c}{J}{}^{\lambda\mu\nu} + \underset{st}{m}{}^{\lambda\mu\nu} = \underset{c}{J}{}^{\lambda\mu\nu} + \partial_\kappa (x^{[\lambda}A^{\mu]}F^{\nu\kappa}). \tag{1.18}$$

Unfortunately, the energy-momentum tensor $\underset{st}{T}{}^{\lambda\mu}$ (1.17) is obviously invalid, as well as the canonical energy-momentum tensor (1.4). So, the (Belinfante-Rosenfeld) procedure [7, 8] is unsuccessful, and the tensors (1.17), (1.18) are never used. But the worst thing is found out when calculating of the standard spin tensor $\underset{st}{\Upsilon}{}^{\lambda\mu\nu} = \underset{c}{\Upsilon}{}^{\lambda\mu\nu} + \underset{st}{s}{}^{\lambda\mu\nu}$, where the spin addend is

$$\underset{st}{s}{}^{\lambda\mu\nu} = \underset{st}{m}{}^{\lambda\mu\nu} - 2x^{[\lambda}\underset{st}{t}{}^{\mu]\nu} = -\partial_\kappa (x^{[\lambda}\tilde{\Upsilon}^{\mu]\nu\kappa}) + x^{[\lambda}\partial_\kappa \tilde{\Upsilon}^{\mu]\nu\kappa} = -\delta_\kappa^{[\lambda}\tilde{\Upsilon}^{\mu]\nu\kappa} = 2\delta_\kappa^{[\lambda}A^{\mu]}F^{\nu\kappa} = -2\delta_\kappa^{[\lambda}A^{\mu]}F^{\kappa\nu} = -2A^{[\mu}F^{\lambda]\nu}$$
$$= 2A^{[\lambda}F^{\mu]\nu} = -\underset{c}{\Upsilon}{}^{\lambda\mu\nu} \tag{1.19}$$

So, we see the procedure gives a standard spin tensor which equals zero! I.e. the procedure eliminates classical spin at all:

$$\underset{st}{\Upsilon}{}^{\lambda\mu\nu} = \underset{c}{\Upsilon}{}^{\lambda\mu\nu} + \underset{st}{s}{}^{\lambda\mu\nu} = 0. \tag{1.20}$$

That is why a spin term is absent in Eq. (1.22).

Note that the addends $\underset{st}{t}{}^{\lambda\mu}$ & $\underset{st}{s}{}^{\lambda\mu\nu}$, though they are unsuitable, satisfy an important equation

$$\partial_\nu \underset{st}{s}^{\lambda\mu\nu} = \underset{st}{t}^{[\lambda\mu]} . \quad (1.21)$$

In spite of the fact that the standard spin tensor is zero, physicists understand they cannot shut eyes on existence of the classical electrodynamics' spin. And they proclaim spin is *in* the moment of the momentum (1.3). I.e., the moment of momentum represents the total angular momentum: orbital angular momentum plus spin. I.e., equation (1.3) encompasses both the spin and orbital angular momentum density of a light beam [2, 4, 9 - 12]:

$$J^{ij} = L^{ij} + S^{ij} = \int_V 2x^{[i}T^{j]0} dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV . \quad (1.22)$$

In the end, it is important to point out that an addition of any term to an energy-momentum tensor, including the addition of a divergence-free term like $-\partial_\nu \underset{c}{\tilde{\Upsilon}}^{\lambda\mu\nu}/2$ (see, e.g. [12, (3.36)]), changes the energy-momentum distribution and changes the total 4-momentum of the system when the field does not change. Really, it is easy to express the energy-momentum tensor of an uniform ball of radius R in the form of $\partial_\nu \Psi^{\lambda\mu\nu}$.

$$\Psi^{00i} = -\Psi^{0i0} = \varepsilon x^i/3 \ \ (r<R), \quad \Psi^{00i} = -\Psi^{0i0} = \varepsilon R^3 x^i/3r^3 \ \ (r>R) \quad (1.23)$$

give

$$T^{00} = \partial_i \Psi^{00i} = \varepsilon \ \ (r<R), \quad T^{00} = \partial_i \Psi^{00i} = 0 \ \ (r>R). \quad (1.24)$$

2. Electrodynamics' spin tensor exists

Contrary to the Belinfante-Rosenfeld procedure, which eliminates spin, we modify the invalid canonical tensors (1.4) – (1.6) by another way [13 - 19]. In contrast to the procedure [7, 8], we use other addends to the canonical energy-momentum and spin tensors. Our addends are

$$t^{\lambda\mu} = \partial_\nu A^\lambda F^{\mu\nu} , \quad (2.1)$$

$$s^{\lambda\mu\nu} = 2A^{[\lambda}\partial^{\mu]}A^\nu , \quad (2.2)$$

instead of (1.15), (1.19). $t^{\lambda\mu}$ gives the Maxwell tensor (1.1)

$$T^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} + \partial_\nu A^\lambda F^{\mu\nu} , \quad (2.3)$$

and $s^{\lambda\mu\nu}$ is obtained from the equation

$$\partial_\nu s^{\lambda\mu\nu} = t^{[\lambda\mu]} , \quad (2.4)$$

which is analogous to (1.21). As a result, we arrive at a quantity

$$2A^{[\lambda}\partial^{|\nu|}A^{\mu]} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + 2A^{[\lambda}\partial^{\mu]}A^\nu , \quad (2.5)$$

instead of the zero, and, at long last, at our spin tensor:

$$\Upsilon^{\lambda\mu\nu} = A^{[\lambda}\partial^{|\nu|}A^{\mu]} + \Pi^{[\lambda}\partial^{|\nu|}\Pi^{\mu]} . \quad (2.6)$$

Here A^λ and Π^λ are magnetic and electric vector potentials which satisfy $\partial_\lambda A^\lambda = \partial_\lambda \Pi^\lambda = 0$, $2\partial_{[\mu}A_{\nu]} = F_{\mu\nu}$, $2\partial_{[\mu}\Pi_{\nu]} = -e_{\mu\nu\alpha\beta}F^{\alpha\beta}$ where $F^{\alpha\beta} = -F^{\beta\alpha}$, $F_{\mu\nu} = F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ is the field strength tensor of a free electromagnetic field. It is evident that the conservation law, $\partial_\nu \Upsilon^{\lambda\mu\nu} = 0$, is held for a free field.

In other words, we introduce a spin tensor $\Upsilon^{\lambda\mu\nu}$ into the modern electrodynamics, i.e. we complete the electrodynamics by introducing the spin tensor, i.e. we claim the total angular momentum consists of the moment of momentum (1.3) *and* a spin term, equation (1.22) is wrong, the moment of momentum (1.3) does not contain spin at all and, in reality,

$$J^{ij} = L^{ij} + S^{ij} = \int_V (2x^{[i}T^{j]0} + \Upsilon^{ij0})dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV + \int_V \Upsilon^{ij0}dV . \quad (2.7)$$

The difference between our statement (2.7) and the common equation (1.22) is verifiable. The cardinal question is, what angular momentum flux, i.e. torque $\tau = dJ/dt$, does a circularly polarized light beam of power P without an azimuth phase structure carry? The common answer, according to (1.22), is

$$\tau = dJ/dt = \mathsf{P}/\omega ; \quad (2.8)$$

our answer, according to (2.7), is

$$\tau = dJ/dt = 2\mathsf{P}/\omega . \quad (2.9)$$

Statements (2.8) & (2.9) are also valid in the case of plane waves or a beam which is much larger than the particle under action if P is the power absorbed by the particle.

3. Theoretical confirmation of (2.9)

3.1. Consider a wide circularly polarized light beam, which is absorbed by a black plane. According to the standard electrodynamics, i.e. to (1.22), tangential forces, which provide angular momentum acting on the plane, act only in the region of the beam surface [1, 6, 9]. However, it is obviously that a couple acts on any small area of the central alight zone of the plane because the plane absorbs a spin flux density. So, according to the conservation law, the edge of any small area must experience compensative tangential forces from the rest of the surface. These tangential stress in the central alight zone is beyond the standard electrodynamics. Only spin tensor (2.6) provides this stress [19].

3.2. To verify statements (2.8), (2.9), we use the angular momentum conservation law. We have calculated the torque acting on a dielectric absorbing the circularly polarized light beam. We use the standard formula

$$\tau = \left| \int [\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B}) + \mathbf{P}\times\mathbf{E}]dV \right| \quad (3.1)$$

[see, for example, [10] eqns. (5.1) & (7.18)]. Here $\mathbf{P} = (\varepsilon - 1)\mathbf{E}$ is the polarization, $\mathbf{j} = \partial_t \mathbf{P}$ is the displacement current, $\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B})$ is the moment of the total Lorentz force per unit volume, and $\mathbf{P}\times\mathbf{E}$ is the torque on electric dipoles per unit volume [20]. The point is the accurate calculation gives the torque (2.9), $\tau = 2\mathsf{P}/\omega$, [16]. At that, we have had for the first two terms and for the last term

$$\left| \int [\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B})]dV \right| = \left| \int \mathbf{P}\times\mathbf{E}\, dV \right| = \mathsf{P}/\omega . \quad (3.2)$$

Loudon [10] calculated the torque exerted by a light beam on a dielectric as well. He used formula (3.1) as well, and he obtained

$$\left| \int [\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B})]dV \right| = \mathsf{P}/\omega \quad (3.3)$$

[see his formulae (7.19) – (7.24)]. But he omitted $\mathbf{P}\times\mathbf{E}$ term without explanations, and P/ω was his finish result for the torque. Taking into account the $\mathbf{P}\times\mathbf{E}$ term, he must obtain our result $2\mathsf{P}/\omega$ (2.9), (3.1).

It is important to note [19] that the central part of the beam produces a torque at the central region of the dielectric due to the spin of the beam,

$$\tau_{\text{spin}} = \left| \int \mathbf{P}\times\mathbf{E}\, dV \right| = \mathsf{P}/\omega , \quad (3.4)$$

and the wall of the beam produces an additional torque due to the orbital angular momentum of the beam.

$$\tau_{\text{orbit}} = \left| \int [\mathbf{r}\times(\mathbf{P}\cdot\nabla)\mathbf{E} + \mathbf{r}\times(\mathbf{j}\times\mathbf{B})]dV \right| = \mathsf{P}/\omega . \quad (3.5)$$

4. Experimental confirmation

4.1. The work of Simpson at al. [21] rather confirms our result (2.7), (2.9) as well. The authors trapped ~2-μm-diameter Teflon particles by a $\mathrm{LG}_{p=0}^{l=1}$ beam of λ = 1047 nm and power P = 25 mW. If the $\mathrm{LG}_{p=0}^{l=1}$ beam is linearly polarized, it carries an orbital angular momentum flux of $\mathsf{P}/\omega = 1.4\cdot 10^{-17}$ J. In this case the trapped particles were rotated with the rotational rate $\Omega = 13/\sec$, according to Fig. 2 from [21] (our Fig.1). This implies that the torque on the particles was $\tau = 8\pi\eta r^3\Omega = 3.3\cdot 10^{-19}\,\mathrm{J} = 0.023\mathsf{P}/\omega$ (formula (3) from [21], here $\eta = 10^{-3}$ kg/m sec is the viscosity, $r = 10^{-6}$ m is the particle radius), and the authors suggested that the particle absorbed about ~2.3% of the power, i.e. $\Delta\mathsf{P} = 0.023\mathsf{P}$. However, this conclusion probably needs to

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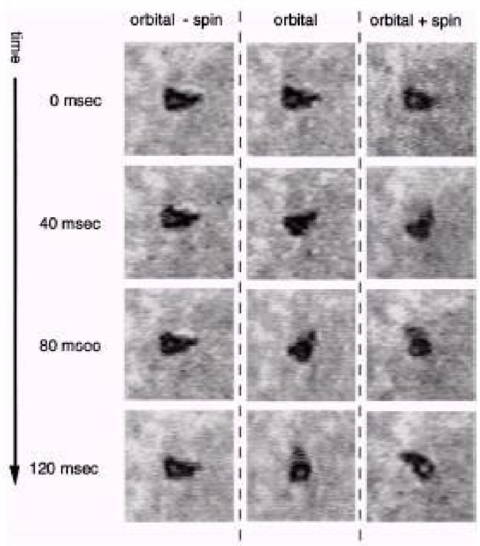

Fig. 2. Successive frames of the video image showing the stop–start behavior of a 2-μm-diameter Teflon particle held with the optical spanner.

Fig. 1. It is from [21]
The particle rotates through 90 degrees during 120 msec

be corrected. The point is a Laguerre-Gaussian beam can exert a torque on particles not only when absorbing, but also when being converted into Hermite-Gaussian beams.

Allen et al. show that a torque exerts on a converter of a Laguerre-Gaussian beam when converting (FIG. 1 from [22], our Fig. 2) because the converter change the phase difference between the Hermite-Gaussian modes that constitute the Laguerre-Gaussian beam (see Figure 13 from [23]). Because the particles had an irregular form, and because ~98% of $LG_{p=0}^{l=1}$ beam passed through the particles in the experiment, it was inevitably that a part of the $LG_{p=0}^{l=1}$ beam was converted into HG modes. If this part was at least 1.2%, the absorption of $\Delta P = 0.012P$ only, instead of 2.3%, could provide the torque $\tau = 3.3 \cdot 10^{-19}\,J$.

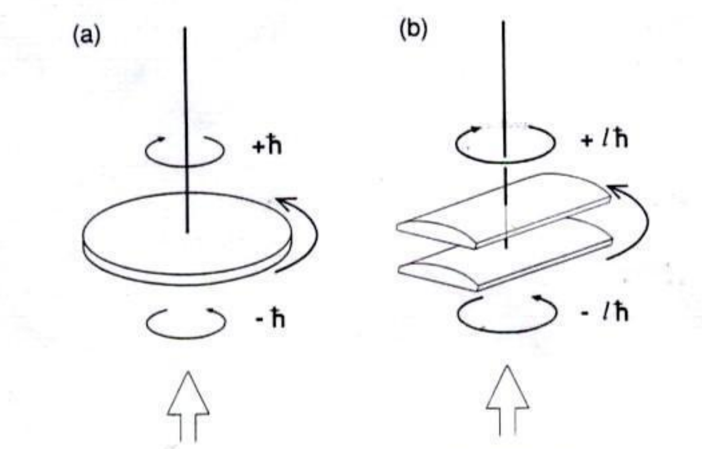

FIG. 1. (a) A suspended $\lambda/2$ birefringent plate undergoes torque in transforming right-handed into left-handed circularly polarized light. (b) Suspended cylindrical lenses undergo torque in transforming a Laguerre-Gaussian mode of orbital angular momentum $-l\hbar$ per photon, into one with $+l\hbar$ per photon.

Fig. 2. It is from [22]
A converter undergoes torque in transforming a Laguerre-Gaussian mode into Hermite-Gaussian one as well

The main point of the Simpson's experiment [21] was a cessation of rotating of the particles when the linearly polarized $LG_{p=0}^{l=1}$ beam became a circularly polarized one if the handedness was opposite to the rotation sense. Thus, we must conclude that the torque associated with the circular polarization equals $2\Delta P/\omega$ because $\tau = 3.3 \cdot 10^{-19}\,J = 2 \cdot 0.012P/\omega$. In any case, because of the possible $LG \rightarrow HG$ conversion, we must conclude that the angular momentum flux related with the circular polarization is larger than P/ω.

4.2. The recent work [24] confirms rather the formulae (2.7), (2.9) as well. In this work a linearly polarized $LG_{p=0}^{l=2}$ beam of λ = 1064 nm and power of P = 20 mW rotates a trapped polystyrene elongated particle with

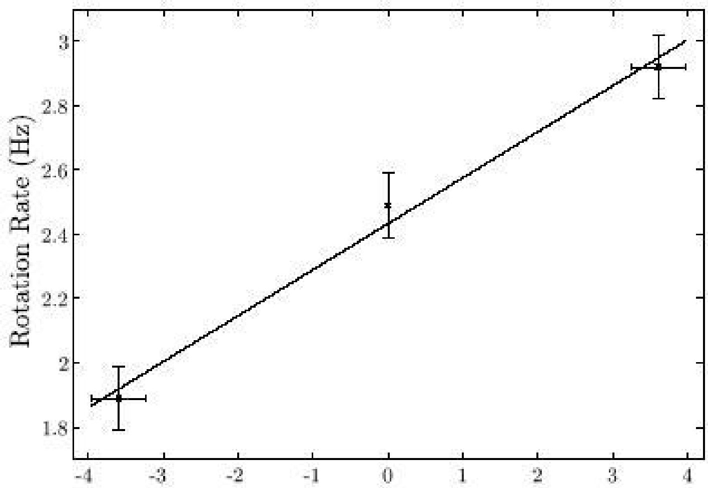

Fig. 3. It is Figure 3 from [24]
The rotational rate of the trapped particle as a function of the degree of circular polarization.
0 corresponds to linear polarization;
4 corresponds to circular polarization.

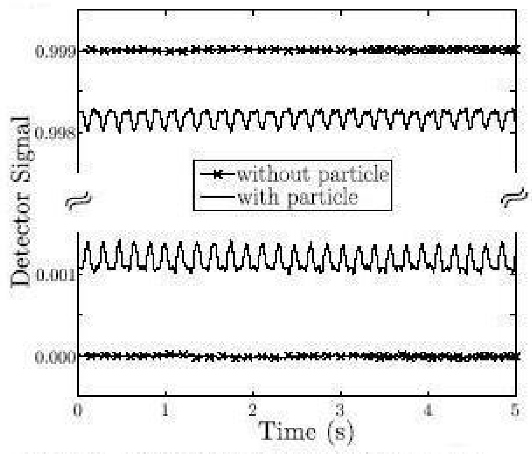

Fig. 4. It is Figure 2 (b) from [24]
Typical signals from photo-detectors 1 and 2 for the case when no particle is trapped and also the case when the particles are trapped by the optical tweezers. The rotational rate of the particle is 3.0 Hz.

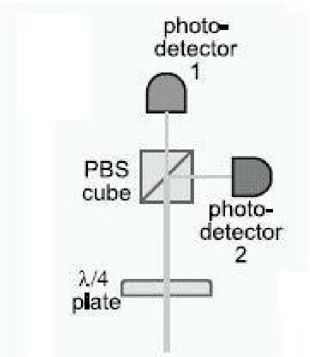

Fig. 5. It is a fragment of Figure 1 from [24]
The system records the right and left circularly polarized constituents of the beam

the rotational rate $\Omega_1 = 2.5$ Hz, and, when circularly polarized, the beam rotates the particle with $\Omega_2 = 3.0$ Hz, according to Figures 3 and 2 from [24] (see our Fig 3 and 4). This increase in the angular velocity, $\Delta\Omega$ = $2\pi\,0.5$/sec, causes the corresponding increase in the drag torque acting on the rotating particle (formula (3) from [24]):
$\Delta\tau = 12\pi\eta a^3\Delta\Omega = 1.2\cdot 10^{-19}$ J (here $a = 10^{-6}$ m is the particle parameter). On the other hand, the increase in the drag torque is provided with change in the degree of circular polarization σ of the beam as the beam passes through the particle. This change is determined by signals of photo-detectors 1 and 2 (see the fragment of Figure 1 from [24] in our Fig. 5.).

The point is an elliptically polarized beam consists of right and left circularly polarized constituents. The electrical field may have the form

$$\mathbf{E} = \exp(ikz - i\omega t)[\alpha(\mathbf{x} + i\mathbf{y}) + \beta(\mathbf{x} - i\mathbf{y})]E_0/\sqrt{2}\,, \qquad (4.1)$$

where $\alpha E_0/\sqrt{2}$ and $\beta E_0/\sqrt{2}$ are the amplitudes of the circularly polarized constituents. The degree of circular polarization of the beam is defined as

$$\sigma = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}\,. \qquad (4.2)$$

To determine α and β, the authors send the beam to a circular polarization detection system consisting of the $\lambda/4$ plate, the polarizing beam splitter cube, and the photo-detectors. The $\lambda/4$ plate converts a circularly polarized constituent to a linearly polarized constituent by introducing $\pi/2$ phase shift of *y*-components, i.e. by multiplying the *y*-components in (4.1) by *i*.

$$\mathbf{E} = \exp(ikz - i\omega t)[\alpha(\mathbf{x} + i\mathbf{y}) + \beta(\mathbf{x} - i\mathbf{y})]E_0/\sqrt{2} \quad\rightarrow\quad \mathbf{E} = \exp(ikz - i\omega t)[\alpha(\mathbf{x} - \mathbf{y}) + \beta(\mathbf{x} + \mathbf{y})]E_0/\sqrt{2}\,. \quad (4.3)$$

According to Fig. 4, the input polarization is 1, and the output polarization is 0.9982 – 0.0012 = 0.997. I.e. $\Delta\sigma$ = 0.003. These results mean that

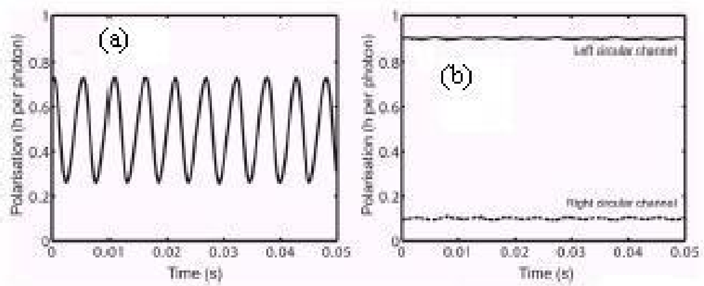

Fig. 6. It is FIG. 2 from [11]
(a) A signal from a photo-detector from which the rotation rate of the particle, 94 Hz, was found.
(b) Signals from the circular polarization detection system

$\Delta\sigma P/\omega \cong 0.3\cdot 10^{-19}$ J (here P = 20 mW and $\omega = 2\pi c/\lambda = 1.9\cdot 10^{15}$/sec). So, we have, according to [24], $\Delta\tau \cong 4\Delta\sigma P/\omega$ instead of $\Delta\tau = 2\Delta\sigma P/\omega$, according to eqn. (2.9), and instead of $\Delta\tau = \Delta\sigma P/\omega$, according to eqn. (2.8). This sizeable polarization contribution to the total torque confirms our statement (2.9).

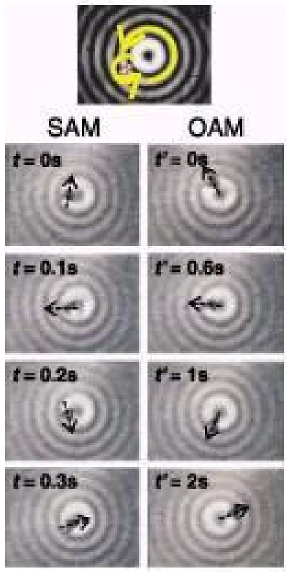

FIG. 1 (color online). A birefringent particle trapped in the first ring of a HOBB rotates simultaneously (i) around its own axis (due to SAM) and (ii) around the beam's axis (due to OAM). The frames were taken from a video at the time indicated in each box.

Fig. 7. It is from [25]
SAM means spin angular momentum, but we contend that the rotation around own axis is caused by spin and angular momentum of the circular polarization.

4.3. In the work [11], a pure Gaussian circularly polarized beam $(\lambda = 1064\text{nm})$ rotates a trapped birefringent particle with $\Omega = 94 \cdot 2\pi = 590/\text{sec}$ when the output polarization of the beam is $\sigma = 0.9 - 0.1 = 0.8$, i.e. $\Delta\sigma = 0.2$. FIG. 2 from [11] (our Fig.6) shows this, but the radius of the particle and the power of the beam are not given in the paper. However, as one can understand from the text and from FIG. 3 of [11], the radius was $r = 1.2\,\mu\text{m}$ and the power was $\mathsf{P} = 100$ mW. From this assumption we get $\Delta\sigma\mathsf{P}/\omega = 1.1\cdot 10^{-17}$ J and $\tau = 8\pi\eta r^3\Omega = 2.6\cdot 10^{-17}$ J. So, $\tau = 2.4\Delta\sigma\mathsf{P}/\omega$, which is rather in accordance with (2.9).

4.4. We are interested in works that show how a trapped particle rotates simultaneously around its own axis and around the beam's axis (due to orbital angular momentum). So we consider the paper [25]. As is shown in FIG. 1 of the paper (our Fig. 7), a particle of a radius approx $r = 1\,\mu$m rotates around its own axis with rotational rate $\Omega_{\text{own}} = 18/\text{sec}$ and around the beam's axis with rotational rate $\Omega_{\text{orbit}} = 2.4/\text{sec}$ along a circle of radius $R = 2.9\,\mu\text{m}$. The beam is a circularly polarised high-order J_2 Bessel beam (HOBB) of $l = 2$. The azimuthal component of the linear momentum density, $p_\phi = \omega l u^2/R$ (formula (2) of [25]), yields the azimuthal force on the particle of $F_\phi = \omega l u^2 \pi r^2/R$ (we set $\varepsilon_0 = c = 1$). But, according to the Stokes's law, $F_\phi = 6\pi\eta r v$. So we have $\omega l u^2\pi r^2/R = 6\pi\eta r v$ and

$$v/R = \Omega_{\text{orbit}} = \omega l u^2 r/6\eta R^2. \tag{4.4}$$

At the same time, z-component of the Poynting vector is $\omega^2 u^2$. So, the power impinging on the particle is $\mathsf{P} = \omega^2 u^2 \pi r^2$. If we use formula (3) from [21], $\tau = 8\pi\eta r^3\Omega_{\text{own}}$, we can obtain $\frac{\tau}{\mathsf{P}/\omega} = \frac{8\pi\eta r^3\Omega_{\text{own}}}{\omega u^2\pi r^2}$. By the use of (4.4) we arrive at

$$\frac{\tau}{\mathsf{P}/\omega} = \frac{4l\Omega_{\text{own}} r^2}{3\Omega_{\text{orbit}} R^2} = 2.3 \tag{4.5}$$

that confirms our formula $\tau = 2\mathsf{P}/\omega$.

4.5. Authors of the interesting work [26] also deal with probe particles, which rotates around their own axises and around the beam's axis. Unfortunately, this work is not quantitative one. Nevertheless, this work confirms an extremely sizeable contribution from the circular polarization of a beam. The authors watched a rotation of a calcite fragment around its own axis due to σ and could not observe this fragment orbiting though they used a Laguerre-Gaussian beam of $l = 8 \quad (\text{LG}^{l=8}_{p=o})$.

5. Supplement

The theoretical confirmation 3.1 may be simplified. Consider a very simple one-dimensional example. Let a rod experience a distributed torque because of applying a set of couples τ (Fig. 8). It is evident that any piece of the rod experiences forces $F = \Delta\tau/\Delta x$ acting on ends of the piece. So a constant shear stress is in the rod as well as the constant stress is in the central alight zone [19]

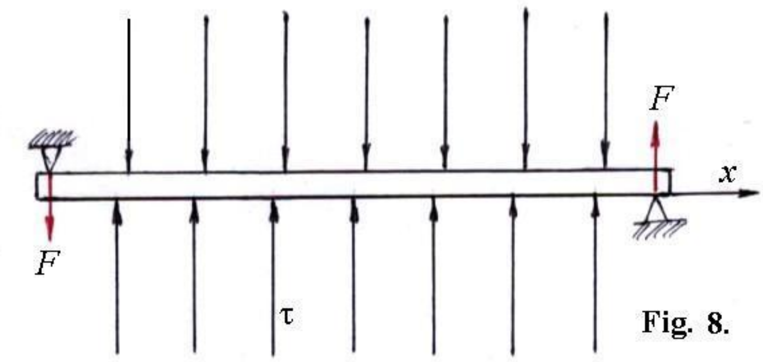

Fig. 8.

The rod experience a distributed torque because of applying a set of couples τ. It is evident that any piece of the rod experiences forces $F = \Delta\tau/\Delta x$ acting on ends of the piece. So a constant shear stress is in the rod

6. Conclusions and Acknowledgements

This paper conveys new physics. We review existing works concerning electrodynamics spin and indicate that existing theory is insufficient to solve spin problems because spin tensor of the modern electrodynamics is zero. Then we show how to resolve the difficulty by introducing a true electrodynamics spin tensor. Our spin tensor doubles a predicted angular momentum of a circularly polarized light beam without an azimuth phase structure. The tensor is needed, in particular, for understanding of a rotating dipole radiation [18] and of mechanical action of a circularly polarized beam [19].

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REFERENCES

1. Heitler, W.: *The Quantum Theory of Radiation.* Clarendon, Oxford (1954), p. 401.
2. Rohrlich, F.: *Classical Charged Particles*. Addison-Wesley, Mass. (1965)
3. Khrapko, R. I.: Moment of the Poynting vector is not spin. V. 2 (poynting_vector_khrapko.pdf). http://www.sciprint.org (2006)
4. Zambrini, R., Barnett, S. M.: Local transfer of optical angular momentum to matter. J. Mod. Optics **52** 1045 (2005)
5. Barut, A. O.: *Electrodynamics and Classical Theory of Particles and Fields.* New York., Macmillan, (1964)
6. Soper, D. E.: *Classical Field Theory*. N.Y., John Wiley (1976).
7. Belinfante, F. J.: Physica 6, 887 (1939).
8. Rosenfeld, L.: Sur le Tenseur D'Impulsion-Energie. Memoires de l'Academie Royale des Sciences de Belgiques **8** No 6 (1940).
9. Jackson, J. D.: *Classical Electrodynamics*. John Wiley (1999), p. 350.
10. Loudon, R.: Theory of the forces by Laguerre-Gaussian light beams on dielectrics. Phys. Rev. A68, 013806 (2003)
11. Bishop, A. I., Nieminen, T. A., Heckenberg, N. R. and Rubinsztein-Dunlop, H.: Optical Microrheology Using Rotating Laser-Trapped Particles. Phys. Rev. Lett. **92**, 198104 (2004).
12. Ryder, L. H.: Quantum Field Theory. Cambridge (1985)
13. Khrapko, R. I.: True energy-momentum tensors are unique. Electrodynamics spin tensor is not zero, physics/0102084
14. Khrapko, R. I.: Experimental verification of Maxwellian electrodynamics. Measurement Techniques, **46**, No. 4, 317-321 (2003)
15. Khrapko, R. I.: Classical spin in space with and without torsion. Gravitation & Cosmology, **10**, No. 1-2, 91-98 (2004)
16. Khrapko, R. I.: Absorption of a circularly polarized beam in a dielectric, *etc.* www.sciprint.org (2006)
17. Khrapko, R. I.: Inevitability of the electrodynamics' spin tensor (Inevitability_spin_khrapko_mar07) www.sciprint.org (2007)
18. Khrapko, R.I.: Radiation of spin by a rotator. mp_arc@mail.ma.utexas.edu REQUEST: send papers NUMBER: 03-315
19. Khrapko, R.I.: Mechanical stresses produced by a light beam. J. Modern Optics, **55**, 1487-1500 (2008)
20. Beth, R. A.: Mechanical detection and measurement of the angular momentum of light" Phys. Rev. **50**, 115 (1936).
21. Simpson, N. B., Dholakia, K., Allen, L. and Padgett, M. J.: Mechanical equivalence of spin and orbital angular momentum of light: an optical spanner. Opt. Lett. **22**, 52-54 (1997)
22. Allen, L., Beijersbergen, M. W., Spreeuw, R. J. C. and Woerdman, J. P.: Orbital angular momentum of light and the transformation of Laguerre-Gaussian modes. Phys. Rev. **A45**, 8185 (1992)
23. Padgett, M., Allen, L.: Light with a twist in its tail. Contemp. Phys. **41**, 275-285 (2000)
24. Parkin, S., Knoner, G., Nieminen, T. A., Heckenberg N. R. and Rubinsztein-Dunlop, H.: Measurement of the total optical angular momentum transfer in optical tweezers. Optics Express **14** 6963 (2006).
25. Garces-Chavez, V., McGloin, D., Padgett, M. J. et al.: Phys. Rev. Lett.. **91**, 093602 (2003).
26. O'Neil, A. T. et al.: Intrinsic and Extrinsic Nature of the Orbital Angular Momentum of a light beam. Phys. Rev. Lett. **88**, 053601 (2002).
27. Khrapko, R. I.: Does plane wave not carry a spin? Amer. J. Phys. 69, 405 (2001).