Foundation of the electrodynamics

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Obviously, the standard Lagrange formalism with the Belinfante-Rosenfeld procedure does not yield the Maxwell energy-momentum tensor and eliminates electrodynamics spin. We change the Belinfante-Rosenfeld addends so that to obtain the Maxwell tensor. As a bonus we are given a spin tensor, which, after a symmetrization in the electro-magnetic sense, turns out to be an electrodynamics spin tensor. Thereby we complete the electrodynamics by introducing the spin tensor. Experimental confirmations of reality of the spin tensor are pointed out.

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1. Introduction

As is well known, photons, i.e. electromagnetic waves, carry spin, energy, momentum, and angular momentum that is a moment of the momentum relative to a given point or to a given axis. Energy and momentum of electromagnetic waves are described by the Maxwell energy-momentum tensor (density)

$$T^{\lambda\mu} = -g^{\lambda\alpha}F_{\alpha\nu}F^{\mu\nu} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4\,, \tag{1}$$

where $F^{\mu\nu} = -F^{\nu\mu}$, $F_{\mu\nu} = F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ is the field strength tensor. For example, $P^i = \int_V T^{i0}dV$ is the momentum of a waves inside the volume *V*, and $dW = \int_a T^{0i}da_i dt$ is the energy that has flowed through the area *a* in the time dt.

An interaction between electromagnetic waves and substance is described by a divergence of the energy-momentum tensor $\partial_\mu T^{\lambda\mu}$, i.e. by the Lorentz force density, viz.,

$$f^\lambda = -\partial_\mu T^{\lambda\mu} = F^{\lambda\beta}\partial^\mu F_{\mu\beta} = j_\beta F^{\lambda\beta}\,. \tag{2}$$

(The Maxwell equations $\partial_{[\lambda}F_{\mu\nu]} = 0$, $\partial^\mu F_{\mu\beta} = j_\beta$ are used here.)

The angular momentum of a waves inside the volume *V* that is a moment of the momentum can be defined as [1]

$$L^{ij} = \int_V 2x^{[i}T^{j]0}dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV\,, \tag{3}$$

and this construction must be named as an orbital angular momentum. However, the modern electrodynamics has no describing of spin, though a concept of classical spin, which differs from the moment of momentum, is contained in the modern theory of fields as is well known.

Realy, the electrodynamics starts from the canonical Lagrangian [2 (4-111)], $\underset{c}{L} = -F_{\mu\nu}F^{\mu\nu}/4$.

Then, by the Lagrange formalism, the canonical energy-momentum tensor [2 (4-113)]

$$\underset{c}{T}^{\lambda\mu} = \partial^\lambda A_\alpha \frac{\partial \underset{c}{L}}{\partial(\partial_\mu A_\alpha)} - g^{\lambda\mu}\underset{c}{L} = -\partial^\lambda A_\alpha F^{\mu\alpha} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4 \tag{4}$$

and the canonical total angular momentum tensor [2 (4-147)]

$$\underset{c}{J}^{\lambda\mu\nu} = 2x^{[\lambda}\underset{c}{T}^{\mu]\nu} + \underset{c}{Y}^{\lambda\mu\nu} \tag{5}$$

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are obtained. Here

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial \underset{c}{L}}{\partial(\partial_{\nu}A_{\alpha})} = -2A^{[\lambda}F^{\mu]\nu}, \qquad (6)$$

is the canonical spin tensor [2 (4-150)]. Its space component is $\mathbf{E}\times\mathbf{A}$:

$$\underset{c}{\Upsilon}^{ij0} = \mathbf{E}\times\mathbf{A}. \qquad (7)$$

The sense of a spin tensor $\Upsilon^{\lambda\mu\nu}$ is as follows. The component Υ^{ij0} is a volume density of spin. This means that $dS^{ij} = \Upsilon^{ij0}dV$ is the spin of electromagnetic field inside the spatial element dV. The component Υ^{ijk} is a flux density of spin flowing in the direction of the x^k axis. For example, $dS_z/dt = dS^{xy}/dt = d\tau^{xy} = \Upsilon^{xyz}da_z$ is the *z*-component of spin flux passing through the surface element da_z per unit time, i.e. the torque acting on the element.

The sense of a total angular momentum tensor, $J^{\lambda\mu\nu}$, is that $dJ^{\lambda\mu} = J^{\lambda\mu\nu}dV_{\nu} = 2x^{[\lambda}T^{\mu]\nu}dV_{\nu} + \Upsilon^{\lambda\mu\nu}dV_{\nu}$ is the total angular momentum in an element dV_{ν}. The corresponding integral is

$$J^{\lambda\mu} = L^{\lambda\mu} + S^{\lambda\mu} = \int_V 2x^{[\lambda}T^{\mu]\nu}dV_{\nu} + \int_V \Upsilon^{\lambda\mu\nu}dV_{\nu}. \qquad (8)$$

It consists of two terms: the first term involves a moment of momentum and represents an orbital angular momentum; the second term is spin. It must be emphasized that a moment of momentum cannot represent spin. This idea is discussed in the paper [3], which was written in response to [4]

2. Imperfection of the Lagrange formalism

However, the canonical tensors (4), (5), (6) are not electrodynamics tensors. They obviously contradict experiments. For example, consider a uniform electric field:

$$A_0 = -Ex,\ \ A_x = 0,\ \ \partial_{\alpha}A^{\alpha} = 0,\ \ F_{x0} = -F^{x0} = \partial_x A_0 = -E, \qquad (9)$$

where A_{α} is the magnetic vector potential from (4). The canonical energy density (4) is negative:

$$\underset{c}{T}^{00} = g^{00}F_{x0}F^{x0}/2 = -E^2/2. \qquad (10)$$

Another example: consider a circularly polarized plane wave (or a central part of a corresponding light beam),

$$E^x = \cos(z-t),\ E^y = -\sin(z-t),\ B^x = \sin(z-t),\ B^y = \cos(z-t),\ \ A^x = \sin(z-t),\ A^y = \cos(z-t) \qquad (11)$$

(for short we set $k = \omega = 1$). A calculation of components of the canonical spin tensor (6) yields

$$\underset{c}{\Upsilon}^{xy0} = 1,\quad \underset{c}{\Upsilon}^{xyz} = 1,\quad \underset{c}{\Upsilon}^{zxy} = A^x B_x = \sin^2(z-t),\quad \underset{c}{\Upsilon}^{yzx} = A^y B_y = \cos^2(z-t). \qquad (12)$$

This result is absurd because, though $\underset{c}{\Upsilon}^{xy0}$ and $\underset{c}{\Upsilon}^{xyz}$ are adequate, the result means that there are spin fluxes in y & x- directions, i.e. in the directions, which are transverse to the direction of the wave propagation.

An opinion exists that a change of the Lagrangian can help to obtain the Maxwell tensor (1) by the Lagrange formalism. A. Barut [5] presented a series of Lagrangians and field equations in Table 1

Table I

Lagrangians and Equations of Motion for the Most Common Fields

Field	*Lagrangian*	*Field Equations*
Free Electromagnetic Field	$L_{I} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2 - B^2)$	$F^{\mu\nu}{}_{,\nu} = 0$
	$L_{II} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(A^{\mu}{}_{,\mu})^2$	$\Box^2 A_{\mu} = 0$
	$L_{III} = -\frac{1}{2}A^{\mu}{}_{,\nu}A_{\mu}{}^{,\nu}$	$\Box^2 A_{\mu} = 0$
	$L_{IV} = \frac{1}{2}[A_{\nu}F^{\mu\nu}{}_{,\mu} - A_{\nu,\mu}F^{\mu\nu}] + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$	$\Box^2 A_{\mu} = 0$
Electromagnetic Field with an External Current	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_{\mu}j^{\mu}$	$F^{\mu\nu}{}_{,\nu} = -\frac{1}{c}j^{\mu}$

However, A. Barut did not show energy-momentum and spin tensors corresponding to these Lagrangians. So, we add Table 2

Table 2

Electrodynamics' Lagrangians, Energy-Momentum Tensors, and Spin Tensors

Lagrangian	Energy-momentum tensor	Spin tensor
$L_I = \underset{c}{L} = -F_{\mu\nu}F^{\mu\nu}/4$	$T_I^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} = -A_\nu{}^{,\lambda}F^{\mu\nu} + g^{\lambda\mu}F_{\sigma\nu}F^{\sigma\nu}/4$	$\Upsilon_I^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}$
$L_{II} = -F_{\mu\nu}F^{\mu\nu}/4 - (A^\mu{}_{,\mu})^2/2$	$T_{II}^{\lambda\mu} = T_I^{\lambda\mu} - A^{\mu,\lambda}A^\sigma{}_{,\sigma} + g^{\lambda\mu}(A^\sigma{}_{,\sigma})^2/2$	$\Upsilon_{II}^{\lambda\mu\nu} = \Upsilon_I^{\lambda\mu\nu} + 2A^{[\lambda}g^{\mu]\nu}A^\sigma{}_{,\sigma}$
$L_{III} = -A^\mu{}_{,\nu}A_\mu{}^{,\nu}/2$	$T_{III}^{\lambda\mu} = -A_\sigma{}^{,\lambda}A^{\sigma,\mu} + g^{\lambda\mu}A_{\sigma,\rho}A^{\sigma,\rho}$	$\Upsilon_{III}^{\lambda\mu\nu} = 2A^{[\lambda}A^{\mu],\nu}$
$L_V = -F_{\mu\nu}F^{\mu\nu}/4 - A_\sigma j^\sigma$	$T_V^{\lambda\mu} = T_I^{\lambda\mu} + g^{\lambda\mu}A_\sigma j^\sigma$	$\Upsilon_V^{\lambda\mu\nu} = \Upsilon_I^{\lambda\mu\nu}$

It is clear, none of these energy-momentum tensors is the Maxwell tensor. And what is more, none of these tensors differs from the Maxwell tensor by a divergence of an antisymmetric quantity only. In other words, none of these tensors has true divergence (2). A method is unknown to get a tensor with the true divergence in the frame of the standard Lagrange formalism.

A desire for such a tensor led Professor Soper to a mistake [6]. He used Lagrangian L_V, but, instead of the tensor $T_V^{\lambda\mu}$, he arrived at a false tensor [6, (8.3.5) – (8.3.9)]

$$\underset{f}{T}^{\lambda\mu} = T_I^{\lambda\mu} + A^\lambda j^\mu\,, \tag{13}$$

which differs from the Maxwell tensor by a divergence of an antisymmetric quantity:

$$T^{\lambda\mu} - \underset{f}{T}^{\lambda\mu} = \partial_\alpha A^\lambda F^{\mu\alpha} - A^\lambda j^\mu = \partial_\alpha(A^\lambda F^{\mu\alpha})\,. \tag{14}$$

3. Belinfante-Rosenfeld procedure

To draw the tensors nearer to the nature, in the frame of the standard procedure, a specific terms,

$$\underset{st}{t}^{\lambda\mu} = -\partial_\nu \tilde{\Upsilon}^{\lambda\mu\nu}/2 \tag{15}$$

and

$$\underset{st}{m}^{\lambda\mu\nu} = -\partial_\kappa(x^{[\lambda}\tilde{\Upsilon}^{\mu]\nu\kappa})\,, \tag{16}$$

are added to the canonical tensors (4) and (5) [7,8] (here $\tilde{\Upsilon}^{\lambda\mu\nu} \overset{def}{=} \underset{c}{\Upsilon}^{\lambda\mu\nu} - \underset{c}{\Upsilon}^{\mu\nu\lambda} + \underset{c}{\Upsilon}^{\nu\lambda\mu} = -2A^\lambda F^{\mu\nu}$). But this procedure does not give the Maxwell tensor, it gives a standard energy-momentum tensor $\underset{st}{T}^{\lambda\mu}$ and a standard total angular momentum tensor $\underset{st}{J}^{\lambda\mu\nu}$,

$$\underset{st}{T}^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} + \underset{st}{t}^{\lambda\mu} = -\partial^\lambda A_\nu F^{\mu\nu} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4 + \partial_\nu(A^\lambda F^{\mu\nu})\,, \tag{17}$$

$$\underset{st}{J}^{\lambda\mu\nu} = \underset{c}{J}^{\lambda\mu\nu} + \underset{st}{m}^{\lambda\mu\nu} = \underset{c}{J}^{\lambda\mu\nu} + 2\partial_\kappa(x^{[\lambda}A^{\mu]}F^{\nu\kappa})\,. \tag{18}$$

The energy-momentum tensor $\underset{st}{T}^{\lambda\mu}$ (17) is obviously invalid, as well as the canonical energy-momentum tensor (4). So, the (Belinfante-Rosenfeld) procedure [7,8] is unsuccessful, and the tensors (17), (18) are never used. But to make matter worse the procedure gives the standard spin tensor which equals zero! I.e. the procedure eliminates classical spin at all:

$$\underset{st}{\Upsilon}^{\lambda\mu\nu} = \underset{st}{J}^{\lambda\mu\nu} - 2x^{[\lambda}\underset{st}{T}^{\mu]\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + \underset{st}{s}^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + 2A^{[\lambda}F^{\mu]\nu} = 0\,; \tag{19}$$

here

$$\underset{st}{s}^{\lambda\mu\nu} = \underset{st}{m}^{\lambda\mu\nu} - 2x^{[\lambda}\underset{st}{t}^{\mu]\nu} = 2A^{[\lambda}F^{\mu]\nu} \tag{20}$$

is the Belinfante-Rosenfeld addend for the canonical spin tensor.

Well, standard spin tensor (19) is zero, but physicists understand they cannot shut eyes on existence of the classical electrodynamics' spin. And they proclaim spin is *in* the moment of the momentum (3). I.e., the moment of momentum represents the total angular momentum: orbital angular momentum plus spin. I.e., equation (3) encompasses both the spin and orbital angular momentum density of a light beam [1,2,4-8]:

$$J^{ij} = L^{ij} + S^{ij} = \int_V 2x^{[i}T^{j]0}dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV\,. \qquad (21)$$

In addition to the above reasoning, here we make a remark concerning so-called general procedure for the symmetrization of the canonical energy-momentum tensor. It is important to point out that an addition of any term to an energy-momentum tensor, including the addition of a divergence-free term like $-\partial_\nu \underset{c}{\tilde{\Upsilon}}{}^{\lambda\mu\nu}/2$, changes the energy-momentum distribution and even can change the total 4-momentum of the system when the electromagnetic field does not change [9,10]. Really, it is easy to express the energy-momentum tensor of an uniform ball of radius R in the form of $\partial_\nu\Psi^{\lambda\mu\nu}$ ($\Psi^{\lambda\mu\nu} = -\Psi^{\lambda\nu\mu}$).

$$\Psi^{00i} = -\Psi^{0i0} = \varepsilon x^i/3 \;\; (r<R), \quad \Psi^{00i} = -\Psi^{0i0} = \varepsilon R^3 x^i/3r^3 \;\; (r>R) \qquad (22)$$

give

$$T^{00} = \partial_i\Psi^{00i} = \varepsilon \;\; (r<R), \quad T^{00} = \partial_i\Psi^{00i} = 0 \;\; (r>R)\,. \qquad (23)$$

It is evident that the addition of a ball changes the total 4-momentum.

4. Electrodynamics' spin tensor

Contrary to the Belinfante-Rosenfeld procedure, which eliminates spin, we modify the invalid canonical tensors (4) – (6) by another way [10-15]. We use other addends to the canonical energy-momentum and spin tensors. Our addends are

$$t^{\lambda\mu} = \partial_\nu A^\lambda F^{\mu\nu}\,, \qquad (24)$$

$$s^{\lambda\mu\nu} = 2A^{[\lambda}\partial^{\mu]}A^\nu\,, \qquad (25)$$

instead of (15), (20). Note that our addents, as well as the standard addends $\underset{st}{t}{}^{\lambda\mu}$, $\underset{st}{s}{}^{\lambda\mu\nu}$, satisfy an important equation

$$\partial_\nu s^{\lambda\mu\nu} = 2t^{[\lambda\mu]}\,. \qquad (26)$$

In contrast to the procedure [7,8], $t^{\lambda\mu}$ gives the Maxwell tensor (1) at once:

$$T^{\lambda\mu} = \underset{c}{T}{}^{\lambda\mu} + \partial_\nu A^\lambda F^{\mu\nu}\,, \qquad (27)$$

however, $s^{\lambda\mu\nu}$ gives a quantity

$$2A^{[\lambda}\partial^{|\nu|}A^{\mu]} = \underset{c}{\Upsilon}{}^{\lambda\mu\nu} + 2A^{[\lambda}\partial^{\mu]}A^\nu\,, \qquad (28)$$

instead of the zero (19).

The quantity (28) is only a part of the wanted spin tensor. The point is that the standard Lagrange formalism (4) – (6) violates an electro-magnetic symmetry because it uses only magnetic vector potential A_α. However a free electromagnetic field provides also existence of an electric (pseudo) vector potential Π^λ. The magnetic and electric vector potentials satisfy

$$2\partial_{[\mu}A_{\nu]} = F_{\mu\nu}\,, \quad 2\partial_{[\mu}\Pi_{\nu]} = -e_{\mu\nu\alpha\beta}F^{\alpha\beta}\,, \quad \partial_\lambda A^\lambda = \partial_\lambda\Pi^\lambda = 0\,, \qquad (29)$$

where $e_{\mu\nu\alpha\beta}$ is the Levi-Civita antisymmetric tensor density. Thereby we suggest the symmetric expression

$$\Upsilon^{\lambda\mu\nu} = A^{[\lambda}\partial^{|\nu|}A^{\mu]} + \Pi^{[\lambda}\partial^{|\nu|}\Pi^{\mu]} \qquad (30)$$

as the electrodynamics spin tensor. It is evident that the conservation law, $\partial_\nu\Upsilon^{\lambda\mu\nu} = 0$, is held for a free field.

In other words, we introduce a spin tensor $\Upsilon^{\lambda\mu\nu}$ into the modern electrodynamics; i.e. we complete the electrodynamics by introducing the spin tensor; i.e. we claim that the total angular momentum consists of the moment of momentum (3) *and* a spin term, that equation (21) is incorrect, that the moment of momentum (3) does not contain spin at all, that, in reality, the total angular momentum in the volume V equals

$$J^{ij} = L^{ij} + S^{ij} = \int_V (2x^{[i}T^{j]0} + \Upsilon^{ij0})dV = \int_V \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV + \int_V \Upsilon^{ij0}dV\,, \qquad (31)$$

and the angle momentum flux on the area a equals

$$\tau^{ij} = \underset{\text{orb}}{\tau}{}^{ij} + \underset{\text{spin}}{\tau}{}^{ij} = \int_a (2x^{[i}T^{j]k} + \Upsilon^{ijk})da_k = \int_a \mathbf{r}\times(\mathbf{E}\times\mathbf{B})d\mathbf{a} + \int_a \Upsilon^{ijk}da_k\,. \tag{32}$$

The difference between our statement (31) and the common equation (21) is verifiable. A cardinal question is, what angular momentum flux, i.e. torque $\tau = dJ/dt$, does a circularly polarized light beam of power P without an azimuth phase structure carry? The common answer, according to (21), is

$$\tau = dJ/dt = \mathsf{P}/\omega\,. \tag{33}$$

Our answer, according to (31), is

$$\tau = dJ/dt = 2\mathsf{P}/\omega\,. \tag{34}$$

Some calculations, in particular, the calculation of absorption of a circularly polarized light beam in a dielectric, the calculation of a radiation of spin by a rotating electric dipole, as well as numerous experimental works confirm our result (34) (see a review in [11]).

Another manifestation of the spin tensor concerns the mechanical stress that arises in a target absorbing a circularly polarized electromagnetic beam. A stress tensor density $\underset{\wedge}{T}{}^{ij}$ describes this stress. The quantity $\underset{\wedge}{T}{}^{ij}$ is calculated in [11]. The two terms of (32) describe two different torques which equal each other but are exerted in different places. This fact, in our opinion, excludes double-counting of the torque.

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