# Canonical spin tensor is wrong

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The canonical spin tensor of the standard electrodynamics is inadequate. This is shown by the use of a plane electromagnetic wave and a standing electromagnetic wave as examples. An improvement of the tensor by adding of a magnetic term is considered. The true spin tensor is demonstrated.
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## I. INTRODUCTION

As is orthodox, the expression $\mathbf{E}\times\mathbf{A}$ is recognized as a volume density of spin of electromagnetic fields. Here **E** and **A** are the electric field strength and the magnetic vector potential, respectively. For example, Jackson [1] divides the angular momentum of a distribution of electromagnetic fields

$$\mathbf{J}=\int\mathbf{x}\times(\mathbf{E}\times\mathbf{B})dV \tag{1}$$

into a spin and an orbital parts,

$$\mathbf{J}=\int[\mathbf{E}\times\mathbf{A}+E^j(\mathbf{x}\times\nabla)A_j]dV \tag{2}$$

(for short I set $\mu_0=1, c=1$). He wrote, "The first term is sometimes identified with the 'spin' of the photon".

Also Ohanian [2] expresses the angular momentum as a sum of two terms:

$$\mathbf{J}=\int\mathbf{x}\times(E^n\nabla A_n)dV+\int\mathbf{E}\times\mathbf{A}\,dV\,. \tag{3}$$

He wrote, "The first term in Eq. (3) represents the orbital angular momentum, and the second term the spin".

The expression $\mathbf{E}\times\mathbf{A}$ is used by Friese et al. [3] for a plane electromagnetic wave. If $\mathbf{E}=\Re\breve{\mathbf{E}}$, $\breve{\mathbf{E}}=\breve{\mathbf{E}}_0\exp[i(z-t)]$, we have $\breve{\mathbf{A}}=-i\breve{\mathbf{E}}$ because $\mathbf{E}=-\partial_t\mathbf{A}$ (the symbol 'breve' marks complex vectors; for short I set $\omega=1$). The authors[3] wrote, "The angular momentum can be found from the electric field $\breve{\mathbf{E}}$ and its complex conjugate $\overline{\mathbf{E}}$ by integrating over all spatial elements $dV$ giving

$$\breve{\mathbf{J}}=\int\overline{\mathbf{E}}\times\breve{\mathbf{E}}\,dV/2i\text{''} \tag{4}$$

(for short I set the permittivity $\varepsilon=1$).

Unfortunately, authors do not explain that the expression $\mathbf{E}\times\mathbf{A}$ is a component of the canonical spin tensor.

As is well known, [4,5] a search of the electromagnetic energy-momentum and spin tensors starts from the free field canonical Lagrangian,

$$\underset{c}{\mathsf{L}}=-F_{\mu\nu}F^{\mu\nu}/4,\qquad F_{\mu\nu}=2\partial_{[\mu}A_{\nu]},\qquad \mu,\nu,...=0,1,2,3. \tag{5}$$

Using this Lagrangian, by the Lagrange formalism physicists obtain the canonical energy-momentum tensor

$$\underset{c}{T}^{\lambda\mu}=\partial^\lambda A_\alpha\frac{\partial\underset{c}{\mathsf{L}}}{\partial(\partial_\mu A_\alpha)}-g^{\lambda\mu}\underset{c}{\mathsf{L}}=-\partial^\lambda A_\alpha F^{\mu\alpha}+g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4, \tag{6}$$

and the canonical total angular momentum tensor

$$\underset{c}{J}^{\lambda\mu\nu}=2x^{[\lambda}\underset{c}{T}^{\mu]\nu}+\underset{c}{\Upsilon}^{\lambda\mu\nu} \tag{7}$$

where

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$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta_{\alpha}^{\mu]}\frac{\partial \underset{c}{L}}{\partial(\partial_{\nu}A_{\alpha})} = -2A^{[\lambda}F^{\mu]\nu}, \qquad (8)$$

is the canonical spin tensor.

Here $F^{\mu\nu} = -F^{\nu\mu}$, $F_{\mu\nu} = F^{\alpha\beta}g_{\mu\alpha}g_{\nu\beta}$ is the field strength tensor. The sense of its components is

$$F^{0i} = -E^{i}, \quad F_{0i} = E_{i}, \quad F^{ij} = -B^{ij}, \quad F_{ij} = -B_{ij}, \quad B_{k} = B^{ij}e_{ijk}, \quad B^{k} = B_{ij}e^{ijk}, \quad i,\ j,... = 1, 2, 3. \qquad (9)$$

For example,

$$F^{x0} = F_{0x} = E^{x} = E_{x}, \quad F^{xy} = F_{xy} = -B^{z} = -B_{z}. \qquad (10)$$

The component

$$\underset{c}{\Upsilon}^{ij0} = -2A^{[i}F^{j]0} = -2A^{[i}E^{j]} = E^{i}A^{j} - E^{j}A^{i} = \mathbf{E}\times\mathbf{A} \qquad (11)$$

is a volume density of spin. This means that

$$dS^{ij} = \underset{c}{\Upsilon}^{ij0}\,dV \qquad (12)$$

is spin of electromagnetic field inside the spatial element $dV$.

The component

$$\underset{c}{\Upsilon}^{ijk} = -2A^{[i}F^{j]k} = 2A^{[i}B^{j]k} \qquad (13)$$

is a flux density of spin in the direction of the $x^{k}$ axis. For example,

$$\underset{c}{\Upsilon}^{xyz} = 2A^{[x}B^{y]z} = A^{x}B^{yz} - A^{y}B^{xz} = A^{x}B_{x} + A^{y}B_{y}, \qquad (14)$$

and

$$dS_{z} = dS^{xy} = \underset{c}{\Upsilon}^{xyz}\,da_{z} = (A^{x}B_{x} + A^{y}B_{y})da_{z} \qquad (15)$$

is $z$-component of spin passing through the surface element $da_{z}$ per unit time.

In this paper we intend to clear up if the expression (8) is adequate. For this purpose we apply the expression to a plane wave and to a standing wave.

## II. PLANE WAVE

Let a right-circularly polarized electromagnetic plane wave, which propagates in $z$-direction, takes the form

$$E^{x} = \cos(z-t), \quad E^{y} = -\sin(z-t), \quad B^{x} = \sin(z-t), \quad B^{y} = \cos(z-t). \qquad (16)$$

Because $\mathbf{A} = -\int \mathbf{E}dt$, we have

$$A^{x} = \sin(z-t), \quad A^{y} = \cos(z-t), \quad \underset{c}{\Upsilon}^{xy0} = 1, \quad \underset{c}{\Upsilon}^{xyz} = 1. \qquad (17)$$

This result is adequate because the Poynting vector is $E^{x}B^{y} - E^{y}B^{x} = 1$, and the ratio of spin to energy, $S/U = 1/\omega$, holds. But a calculation of other components of the spin tensor yields

$$\underset{c}{\Upsilon}^{zxy} = A^{x}B_{x} = \sin^{2}(z-t), \qquad \underset{c}{\Upsilon}^{yzx} = A^{y}B_{y} = \cos^{2}(z-t). \qquad (18)$$

This result is absurd, because it means that there are spin flux in the direction, which is transverse to the direction of the wave propagation.

## III. STANDING WAVE

Let us take the sum of the wave (16)

$$E_{1}^{x} = \cos(z-t), \quad E_{1}^{y} = -\sin(z-t), \quad B_{1}^{x} = \sin(z-t), \quad B_{1}^{y} = \cos(z-t). \qquad (19)$$

and a wave reflecting off a perfect conductive plane $z = 0$

$$E_{2}^{x} = -\cos(z+t), \quad E_{2}^{y} = -\sin(z+t), \quad B_{2}^{x} = -\sin(z+t), \quad B_{2}^{y} = \cos(z+t). \qquad (20)$$

The total field is

$$E^x = E_1^x + E_2^x = 2\sin z \sin t\,, \qquad E^y = E_1^y + E_2^y = -2\sin z \cos t\,, \tag{21}$$

$$B^x = B_1^x + B_2^x = -2\cos z \sin t\,, \qquad B^y = B_1^y + B_2^y = 2\cos z \cos t\,. \tag{22}$$

Magnetic vector potential, according to $\mathbf{A} = -\int \mathbf{E} dt$, is

$$A^x = 2\sin z \cos t, \quad A^y = 2\sin z \sin t\,. \tag{23}$$

So, we can calculate components of the spin tensor:

$$\underset{c}{\Upsilon}^{xy0} = 4\sin^2 z, \quad \underset{c}{\Upsilon}^{xyz} = 0\,. \tag{24}$$

The result $\underset{c}{\Upsilon}^{xyz} = 0$ is adequate because there is no spin flux to the conductive plane, but $\underset{c}{\Upsilon}^{xy0} = 4\sin^2 z$ raises a doubt because there is no cause of dividing electromagnetic spin into layers. As is known, energy density is constant: $(E^2 + B^2)/2 = 2$.

Unfortunately, the calculation of other components of the spin tensor yields the absurd result as well

$$\underset{c}{\Upsilon}^{zxy} = A^x B_x = -\sin 2z \sin 2t\,, \qquad \underset{c}{\Upsilon}^{yzx} = A^y B_y = \sin 2z \sin 2t\,. \tag{25}$$

## IV. MAGNETIC PART OF SPIN

The canonical spin tensor (8), (11) is obvious not symmetric in the sense of electric - magnetic symmetry. It represents only the electric field, $\mathbf{E}$, $\mathbf{A} = -\int \mathbf{E} dt$. It calls forth the unsatisfactory result (24) for $\Upsilon^{xy0}$. So, it makes sense to symmetrize the spin tensor by adding a term

$$\underset{c.m}{\Upsilon}^{\lambda\mu\nu} = -\Pi_*^{[\lambda} F_*^{\mu]\nu}\,. \tag{26}$$

The point is that the electrodynamics is asymmetric. Magnetic induction is closed, but magnetic field strength has electric current as a source:

$$\partial_{[\lambda} F_{\mu\nu]} = 0, \quad \partial_\nu F^{\mu\nu} = -j^\mu\,. \tag{27}$$

So, a magnetic vector potential $A_\nu$ exists, but, generally speaking, an electric vector potential does not exist. However, when currents are absent the symmetry is restored, and a possibility to introduce an electric multivector potential $\Pi^{\lambda\mu\nu}$ appears. The electric multivector potential satisfies the equation

$$\partial_\nu \Pi^{\lambda\mu\nu} = F^{\lambda\mu}\,. \tag{28}$$

A covariant pseudovector, dual relative to the multivector potential,

$$\Pi_\kappa^* = e_{\kappa\lambda\mu\nu} \Pi^{\lambda\mu\nu}\,, \tag{29}$$

is an analog of the magnetic vector potential $A_\kappa$. We name it the electric vector potential. It is inserted into (26). Pseudotensor $F_*^{\mu\nu}$ is dual to the field strength tensor $F^{\mu\nu}$,

$$F_*^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} e_{\alpha\beta\gamma\delta} F^{\gamma\delta} / 2\,. \tag{30}$$

The sense of its components is

$$F_*^{0i} = B^i, \quad F_*^{ij} = -e^{ijk} E_k\,. \tag{31}$$

For example,

$$F_*^{0x} = B^x, \quad F_*^{yz} = -E_x\,. \tag{32}$$

So, accordingly with (26), omitting $*$ of $\Pi$, we have,

$$\underset{c.m}{\Upsilon}^{ij0} = -\Pi^{[i} F_*^{j]0} = (\Pi^i B^j - \Pi^j B^i)/2 = (\Pi \times \mathbf{B})/2\,. \tag{33}$$

$$\underset{c.m}{\Upsilon}^{ijk} = -\Pi^{[i} F_*^{j]k}\,. \tag{34}$$

For example,

$$\underset{c.m}{\Upsilon}^{xyz} = -\Pi^{[x}F_*^{y]z} = (\Pi^x E_x + \Pi^y E_y)/2\,, \qquad \underset{c.m}{\Upsilon}^{zxy} = -\Pi^{[z}F_*^{x]y} = (\Pi^z E_z + \Pi^x E_x)/2\,. \tag{35}$$

A relation between $\Pi$ and $F$ can be readily obtained in the vector form as follows. If $\mathrm{div}\mathbf{D} = 0$, then $\mathbf{D} = \mathrm{curl}\,\Pi$. If also $\partial\mathbf{D}/\partial t = \mathrm{curl}\mathbf{H}$, then $\mathbf{H} = \partial\Pi/\partial t$, but we set $\mathbf{H} = \mathbf{B}$, so

$$\partial\Pi/\partial t = \mathbf{B}\,. \tag{36}$$

## V. TOTAL SPIN TENSOR

We consider here a total spin tensor corresponding to the canonical spin tensor (8):

$$\underset{tot}{\Upsilon}^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu}/2 + \underset{c.m}{\Upsilon}^{\lambda\mu\nu} = -A^{[\lambda}F^{\mu]\nu} - \Pi_*^{[\lambda}F_*^{\mu]\nu}\,. \tag{37}$$

For the plane wave (16) we have the adequate result as well as (17),

$$\Pi^x = \cos(z-t), \quad \Pi^y = -\sin(z-t), \quad \underset{tot}{\Upsilon}^{xy0} = 1, \quad \underset{tot}{\Upsilon}^{xyz} = 1 \tag{38}$$

But the calculation of other components of the spin tensor yields the absurd result as well (cf. (18)),

$$\underset{tot}{\Upsilon}^{zxy} = (A^x B_x + \Pi^x E_x)/2 = 1/2\,, \qquad \underset{tot}{\Upsilon}^{yzx} = (A^y B_y + \Pi^y E_y)/2 = 1/2\,, \tag{39}$$

however, the magnetic part of spin tensor flattens the layers (18) of spin flux in the direction, which is transverse to the direction of the wave propagation.

For the standing wave (21) - (23) we have, instead of (23), (24), (25),

$$\Pi^x = 2\cos z\cos t, \quad \Pi^y = 2\cos z\sin t\,. \quad \underset{tot}{\Upsilon}^{xy0} = 2, \quad \underset{tot}{\Upsilon}^{xyz} = 0\,. \quad \underset{tot}{\Upsilon}^{zxy} = 0 \qquad \underset{tot}{\Upsilon}^{yzx} = 0 \tag{40}$$

The results are adequate because the energy density is $(E^2 + B^2)/2 = 2$, and the ratio of spin to energy, $S/U = 1/\omega$, holds.

Thus the magnetic part of spin tensor flattens the spin layers and eliminates the transverse spin flux in the case of standing waves. This proves usefulness of adding the magnetic term (26) to the canonical spin tensor (8). Nevertheless the transverse spin flux in the case of plane waves proves that the canonical spin tensor is inadequate even with the magnetic addition.

## VI. TRUE SPIN TENSOR

A new spin tensor was presented and applied in a series of works [6-8]. This tensor is a sum of electric and magnetic terms as well,

$$\Upsilon^{\lambda\mu\nu} = \underset{e}{\Upsilon}^{\lambda\mu\nu} + \underset{m}{\Upsilon}^{\lambda\mu\nu} = A^{[\lambda}\partial^{|\nu|}A^{\mu]} + \Pi^{[\lambda}\partial^{|\nu|}\Pi^{\mu]}. \tag{41}$$

When calculating, we must take account of $\partial^z = g^{zz}\partial_z = -\partial_z$. For the plane wave using (16) yields

$$\Upsilon^{xy0} = 1, \quad \Upsilon^{xyz} = 1, \quad \Upsilon^{zxy} = \Upsilon^{yzx} = 0\,. \tag{42}$$

For the standing wave using (21), (22) yields

$$\Upsilon^{xy0} = 2, \quad \Upsilon^{xyz} = 0, \quad \Upsilon^{zxy} = \Upsilon^{yzx} = 0\,, \tag{43}$$

which was to be demonstrated.

## VII. NOTES and ACKNOWLEDGEMENTS

Expression (41) for the spin tensor was submitted to JETP on Jan. 27, 1999. This result was rejected more than 350 times by scientific journals. For example (I show an approximate number of the rejections in parentheses): JETP Lett. (8), JETP (13), TMP (10), UFN (9), RPJ (70), AJP (14), EJP (4), EPL (5), PRA (2), PRD (4), PRE (2), APP (5), FP (6), PLA (7), OC (2), JPA (4), JPB (1), JMP (4), JOPA (1), JMO (1), CJP (1), OL (1), NJP (2), arXiv (70). In particular, PRA rejected a paper “Beth's experiment modification” submitted on Sun, 16 Nov 2003 06:36:00.

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