Response to Question #79. Does a plane wave carry spin angular momentum?

In a recent contribution to this journal, R. I. Khrapko[1] asks, "Does plane wave not carry a spin?" The question arises because although a circularly polarized light beam might be expected to possess a spin angular momentum of $\hbar$ per photon, were it to do so it would appear to contradict the classical argument that an infinite plane wave carries no angular momentum.[2] The classical argument is that an angular momentum in the direction of propagation can only be produced by a linear momentum in the azimuthal direction; a transverse or azimuthal momentum requires an electric or magnetic field in the propagation direction. This requirement is clearly incompatible with a plane wave that has only transverse electric and magnetic fields.

In the laboratory even very large, uniform amplitude, diameter beams are effectively apertured by the object with which they interact. Any form of aperture introduces an intensity gradient and a detailed analysis using Maxwell's equations shows that a field component is induced in the propagation direction and so the dilemma is potentially resolved.

Khrapko[1] proposes a specific experiment with a two-element absorber comprising an inner disc and a closely fitting outer annulus. His concern is that because there is no intensity gradient between the inner disc and outer annulus, the inner disc experiences no torque. It would follow from the absence of such a torque that the circularly polarized plane wave carries no spin angular momentum.

Our recent review article[3] cited by Khrapko, considers both the spin and orbital angular momentum of light beams. The separation of the angular momentum into spin and orbital contributions, where spin is associated with circular polarization and the orbital contribution is associated with an azimuthal phase structure, is normal in both classical and quantum physics. In his question, Khrapko is concerned with the angular momentum arising from circular polarization, that is, the spin term. In our review, we give a derivation of an expression showing how the local spin angular momentum density per photon is proportional to the radial intensity gradient of a light beam:

$$j_z = -\frac{r}{2}\frac{1}{|u|^2}\frac{\partial |u|^2}{\partial r}\hbar\sigma, \tag{1}$$

where $\sigma=0$ for linearly polarized light and $\sigma=\pm 1$ for right- and left-handed circularly polarized light respectively, $|u|^2$ is the beam intensity, and r is the distance from the axis. For a plane wave there is no gradient and the spin density is zero. In a more recent paper[4] we investigated the paradoxes associated with relationship (1) in considerable detail, particularly for laboratory realizable fields possessing gradients, rather than the idealized plane wave. Our approach echoes that of Simmonds and Gutmann[5] and may be applied to problems of the type raised by Khrapko.[1]

Consider first the simpler problem of a circularly polarized "infinite plane wave" interacting with a suspended absorber

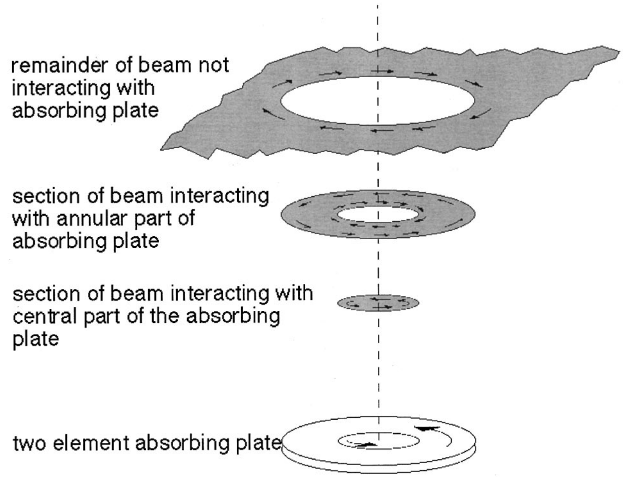

Fig. 1. When suspended in a circularly polarized plane wave, a two element absorbing plate comprising a central disc and outer annulus experiences a torque on both components. The torques arise from the effective aperturing of the light beam, such that the large intensity gradient at the perimeter of the plates results in azimuthal components to the momentum density.

of circular cross section. It is useful to represent the plane wave as the sum of two beams. The first beam has the same diameter as the plate; the rapid falloff in intensity at its edge gives rise to an angular momentum about the axis. The absorption of the beam and its associated angular momentum results in a torque on the plate. The second beam corresponds to the rest of the plane wave and has an equal but opposite transverse momentum content around its inner edge. However, this second beam plays no role as it does not overlap with the plate, is not absorbed, and its polarization remains unchanged. The circumference of the plate scales linearly with its radius and the resulting torque therefore scales with the square of the radius and is, as expected, proportional to the intersected area of the beam. Note that if the absorbing plate is replaced by a transparent quarter wave-plate, essentially the same argument can be applied. In this case, although there is no absorption, the polarization state of the inner beam is transformed to a linear polarization and so the light's angular momentum is transferred to the plate.

Now consider the specific problem raised by Khrapko, namely that of a two-element absorbing plate comprising an inner disc and a close fitting outer annulus. The plane wave must be decomposed into three beams; an inner disc, an intermediate annulus, and the remainder. The inner beam acts on the inner disc of the plate as before, producing a torque proportional to its area. The annular beam acts on the annular section of the plate. Because the intensity gradient is of opposite sign at the inner and outer edges, the resulting torques also have opposite sign. However, the outer edge is longer and acts about a larger radius vector giving a net torque proportional to the area of the annulus and in the same direction as that on the inner disc (see Fig. 1). The third beam again plays no role as it does not interact with the plate. We note that at the join between the disc and annulus, the two beams have equal and opposite azimuthal momenta, giving a total azimuthal momentum of zero, as expected for a plane wave.

This argument may, in principle, be extended to any experimental configuration. Consequently, when interacting with an object or objects of finite extent, a circularly polarized light beam of any extent or intensity distribution can always be considered to be carrying a spin angular momentum of $\pm\hbar$ per photon, as demonstrated by Beth in 1936.[6]

[1] R. I. Khrapko, "Does plane wave not carry a spin?," Am. J. Phys. **69**, 405 (2001).

[2] W. Heitler, *The Quantum Theory of Radiation* (Clarendon, Oxford, 1954), p. 401.

[3] L. Allen, M. Babiker, and M. J. Padgett, "The orbital angular momentum of light," in *Progress in Optics*, edited by E. Wolf (Elsevier, Amsterdam, 1999), Vol. XXXIX.

[4] L. Allen and M. J. Padgett, "The Poynting vector in Laguerre–Gaussian beams and the interpretation of their angular momentum density," Opt. Commun. **184**, 67–71 (2000).

[5] J. W. Simmonds and M. J. Gutmann, *States, Waves and Photons* (Addison–Wesley, Reading, MA, 1970).

[6] R. A. Beth, "Mechanical detection and measurement of the angular momentum of light," Phys. Rev. **50**, 115–125 (1936).

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Answer to Question #79. Does plane wave not carry a spin?

This question about the angular momentum (spin) of the electromagnetic wave[1] is an interesting one because of its fundamental nature, yet it is scarcely discussed in typical textbooks.

Feynman[2] shows the existence of angular momentum in a plane wave by showing that there is angular momentum absorbed when a plane wave moves through a dielectric medium. He utilizes a simplified model that assumes the viscous motion of the electrons. A weak point of his explanation is that the viscosity is assumed implicitly and the phase lag needed for the absorption is introduced without justification. The phase lag really does exist, but it appears entirely due to losses. In a lossless system there is no phase lag and no absorption of either energy or angular momentum.

First, we show that angular momentum is carried by an infinite uniform circularly polarized plane wave and is induced in a lossless system of free electrons even without steady-state absorption, i.e., without being transferred further from the electrons to a macroscopic body. The electric and magnetic fields of a uniform circularly polarized monochromatic plane wave can be represented as $\mathbf{E}=E_0 \exp(-i\omega t)\times(\hat{x}+i\hat{y})$ and $\mathbf{H}=H_0 \exp(-i\omega t)(-i\hat{x}+\hat{y})$, where $H_0=E_0/\eta$ and $\eta=\sqrt{\mu/\epsilon}$. The motion of a free electron is governed by Newton's equation, $m\, d^2\mathbf{r}/dt^2=\mathbf{F}$, where $\mathbf{r}$ is the radius vector of the electron and $\mathbf{F}=e(\mathbf{E}+d\mathbf{r}/dt\times\mu\mathbf{H})$ is the Lorentz force (a small force due to radiation damping is neglected). Let us try the function $\mathbf{r}(t)=A \exp(-i\omega t)(\hat{x}+i\hat{y})+v_0 t\hat{z}$ describing electron motion along a helix as a possible solution to this equation. If we substitute $\mathbf{r}(t)$ and the expressions for $\mathbf{E}$ and $\mathbf{H}$ into the equation of motion, we obtain $-m\omega^2 A \exp(-i\omega t)(\hat{x}+i\hat{y})=eE_0 \exp(-i\omega t)(1-v_0/c)(\hat{x}+i\hat{y})$. We can see that $\mathbf{r}(t)$ is, indeed, a solution if $A=-eE_0(1-v_0/c)/m\omega^2$.

Thus, an electron in the field of a circularly polarized plane wave moves along a helix, that is, it acquires the angular momentum induced by the wave (the angular momentum is induced when the wave is just turned on). In this example, the electron is strongly coupled to the wave and does not interact with any other system. As a result, $\mathbf{r}$ is parallel to $\mathbf{E}$ (A is real), and there is no absorption of angular momentum of the steady-state wave. However, when the electrons are coupled to a medium, this motion will lead to absorption of the angular momentum. Notice that an electron acted upon by the fields of a linearly polarized plane wave does not execute circular motion; in the above relations, the complex vectors $\hat{x}+i\hat{y}$ and $-i\hat{x}+\hat{y}$ are replaced by $\hat{x}$ and $\hat{y}$, respectively.

Now that the existence of the angular momentum is clearly seen, the question arises of how one can represent it in terms of the vectors of the electromagnetic field. A rigorous answer is provided by quantum mechanics so that the question is a matter of an adequate quasiclassical approximation. A useful discussion of the angular momentum using a quasiclassical approximation is given by Simmons and Guttmann.[3] Here we give the classical argument.

The conventional definition of the total angular momentum of the wave beam is[4] $\mathbf{J}=c^{-2}\int_V dv\ \mathbf{r}\times\langle\mathbf{E}\times\mathbf{H}\rangle$, where $\langle\ \rangle$ is the time average. It implicitly assigns the nonzero density of angular momentum only to the border of the beam (this is the only domain where the electric and magnetic fields, due to their decay, have nonvanishing axial components so that the Poynting vector $\mathbf{P}=\langle\mathbf{E}\times\mathbf{H}\rangle$ acquires an azimuthal component,[3] providing nonzero axial component for the cross-product $\mathbf{r}\times\mathbf{P}$). Such an assignment is counterintuitive and raises a series of puzzles as outlined in Ref. 1. Nevertheless, it is consistent from the macroscopic point of view, which does not specify the density distribution. As shown in Ref. 3, this assignment gives a consistent interpretation when considering the absorption of the angular momentum by a small body inside the beam. Indeed, after the absorption, the central part of the beam is absent and the inner border of the beam carries an angular momentum of the opposite sign accounting for the absorption.

There is, however, a second definition of $\mathbf{J}$ which assigns the density of the angular momentum to the inner points of the beam. It is derived[3] by performing an integration by parts over the beam radius which moves the nonzero values of the density of $\mathbf{J}$ from the border to the bulk of the beam. Written in complex variables for a monochromatic wave, the new form of $\mathbf{J}$ is $\mathbf{J}=\int_V dv\ \mathbf{m}$, where $\mathbf{m}$ is the density of the angular momentum defined as $\mathbf{m}=\mathrm{Re}\{\epsilon(\mathbf{E}^*\times\mathbf{E})/2i\omega\}$. One can easily check that $\mathbf{m}$ is locally nonzero even for the infinite uniform plane wave if the latter is circularly polarized, while it is zero for the linearly polarized wave.

This second definition of the density of angular momentum is correct in all aspects although, formally, the solution is not unique because any other equivalent presentation for the integral $\mathbf{J}$ is also acceptable (for example, the one obtained by yet another integration by parts). In view of such an ambiguity, is there any criterion for choosing a unique definition, and why is this second definition claimed to be correct?

The criterion is the requirement of proper relations between the densities of energy and momenta at each point. To illustrate, consider Maxwell's equations for the amplitudes of plane waves, $\hat{k}\times\mathbf{H}_0=-c\epsilon\mathbf{E}_0$ and $\hat{k}\times\mathbf{E}_0=c\mu\mathbf{H}_0$. By cross multiplying them by $\mathbf{E}_0^*$ and $\mathbf{H}_0^*$, respectively, we obtain the equations $\mathbf{E}_0^* \times \mathbf{H}_0 = c(\epsilon \mathbf{E}_0^* \cdot \mathbf{E}_0)\hat{k}$ and $\epsilon(\mathbf{E}_0^* \times \mathbf{E}_0) = -c(\epsilon \mathbf{E}_0^* \cdot \eta \mathbf{H}_0)\hat{k}$. The first equation provides the proper relation between power flux and the energy density. The second provides the relation between angular momentum and the energy density which is proportional to $\epsilon \mathbf{E}_0^* \cdot \eta \mathbf{H}_0$. These relationships are then easily cast into the quantum mechanical form using the concept of photons.

Thus, an alternative representation for the total angular momentum of the electromagnetic wave with the density of angular momentum **m** defined by the second definition resolves all the puzzles concerning the spatial localization of this quantity and secures the correspondence between the quantum and classical formulations.

[1]R. I. Khrapko, "Question #79. Does plane wave not carry a spin?" Am. J. Phys. **69** (4), 405 (2001).
[2]R. P. Feynmann, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA, 1965), Vol. 3, Chap. 17, p. 10.
[3]J. W. Simmons and M. J. Guttmann, *States, Waves and Photons: A Modern Introduction to Light* (Addison–Wesley, Reading, MA, 1970).
[4]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999), pp. 350 and 608.

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The angular part of the Schrödinger equation for the hydrogen atom

In all of the most popular quantum textbooks, the series solution to the harmonic oscillator is shown to break because of the requirements of normalization which lead to the energy eigenvalues. Why do none of these textbooks discuss the necessity for the breaking of the series for the angular part of the Schrödinger equation for the hydrogen atom? In fact, it appears that the series does not have to break for all eigenvalues, $m > 1$.

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