# Angular momentum of the electromagnetic field

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## Abstract
A defect of the standard classical electrodynamics is the theory does not know spin. The spin tensor of the classical electrodynamics is zero. The absence of spin in the theory implies an absurd corollary: (i) a circularly polarized plane wave has no angular momentum at all in direct contradiction to quantum theory; (ii) an orbital angular momentum of a circularly polarized beam without an azimuth phase structure is erroneously recognized as spin.

An expression for electrodynamics' spin tensor is presented.

## 1. Today's state
A recent paper [1] is very symptomatic. The author points out that, on the one hand, "the angular momentum of a classical electromagnetic plane wave is predicted to be, on theoretical grounds, exactly zero", but, on the other hand, "it has been known ever since the experiment of Beth that a circularly polarized plane wave does carry angular momentum". This paradox has been the subject of discussion for a long time, and now the author has taken part in the discussion. But I think this discussion is hopeless because a defect of the modern field theory is the cause of the contradiction.

The defect is the standard classical electrodynamics does not know spin. The spin tensor of the classical electrodynamics equals zero. The well known canonical spin tensor,

$$\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha}\frac{\partial \underset{c}{\mathcal{L}}}{\partial(\partial_\nu A_\alpha)} = -2A^{[\lambda}F^{\mu]\nu}, \quad \underset{c}{\Upsilon}^{ij0} = \mathbf{E}\times\mathbf{A}, \tag{1}$$

is not an electrodynamics spin tensor. The canonical energy-momentum tensor

$$\underset{c}{T}^{\lambda\mu} = \partial^\lambda A_\alpha \frac{\partial \underset{c}{\mathcal{L}}}{\partial(\partial_\mu A_\alpha)} - g^{\lambda\mu}\underset{c}{\mathcal{L}} = -\partial^\lambda A_\alpha F^{\mu\alpha} + g^{\lambda\mu}F_{\alpha\beta}F^{\alpha\beta}/4 \tag{2}$$

and the canonical total angular momentum tensor

$$\underset{c}{J}^{\lambda\mu\nu} = 2x^{[\lambda}\underset{c}{T}^{\mu]\nu} + \underset{c}{\Upsilon}^{\lambda\mu\nu}. \tag{3}$$

are not an electrodynamics tensors as well.

Physicists modify these tensors. They add specific terms to the canonical tensors [2, 3] and arrive to the standard energy-momentum tensor $\Theta^{\lambda\mu}$, the standard total angular momentum tensor $\underset{st}{J}^{\lambda\mu\nu}$, and the standard spin tensor $\underset{st}{\Upsilon}^{\lambda\mu\nu}$, which is zero:

$$\Theta^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} - \partial_\nu \underset{c}{\tilde{\Upsilon}}^{\lambda\mu\nu}/2 = -F^{\lambda\sigma}F^{\mu\kappa}g_{\sigma\kappa} + g^{\lambda\mu}F_{\sigma\kappa}F^{\sigma\kappa}/4 + A^\lambda\partial_\sigma F^{\mu\sigma}, \quad \underset{c}{\tilde{\Upsilon}}^{\lambda\mu\nu} \overset{def}{=} \underset{c}{\Upsilon}^{\lambda\mu\nu} - \underset{c}{\Upsilon}^{\mu\nu\lambda} + \underset{c}{\Upsilon}^{\nu\lambda\mu}, \tag{4}$$

$$\underset{st}{J}^{\lambda\mu\nu} = \underset{c}{J}^{\lambda\mu\nu} - \partial_\kappa(x^{[\lambda}\underset{c}{\tilde{\Upsilon}}^{\mu]\nu\kappa}), \tag{5}$$

$$\underset{st}{\Upsilon}^{\lambda\mu\nu} = \underset{st}{J}^{\lambda\mu\nu} - 2x^{[\lambda}\Theta^{\mu]\nu} = 0. \tag{6}$$

The absence of spin in the standard electrodynamics implies an absurd corollary: a circularly polarized plane wave has no angular momentum at all in direct contradiction to quantum theory. In accordance with this absurdity, the author [1] uses mystical concepts of "angular momentum in an *actual* form" and "angular momentum in an *potential* form".

The absence of spin in the frame of the modern classical electrodynamics means that the angular momentum given by the well known formula $\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV$ is an *orbital* angular momentum,

$$\mathbf{L} = \int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV . \tag{7}$$

This integral is zero in the important cases:
(1) $\mathbf{L}=0$ for a plane wave because $\mathbf{E}\times\mathbf{B}$ is parallel to the direction of propagation,
(2) $\mathbf{L}=0$ for the Beth experiment [4] because $\mathbf{E}\times\mathbf{B}=0$. In the Beth experiment a beam of circularly polarized light exerted a torque on a doubly refracting plate, which changes the state of polarization of the light beam. But, it is evident that the Poynting vector equals zero in the experiment because the passed beam is added with the reflected one. Therefore the result of the Beth experiment cannot be understood in the frame of the standard electrodynamics without spin.

For a circularly polarized beam without an azimuth phase structure the contribution to the integral (7) arises from the skin of the beam where **E** and **B** fields have a component parallel to the wave vector (the field lines are closed loops) and the mass-energy whirls around the bulk of the beam [5]. It confirms the orbital character of the angular momentum.

Nevertheless, physicists consider the expression (7) as the total angular momentum of the electromagnetic field. They claim

$$\mathbf{J} = \int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV \tag{8}$$

and try to decompose it into an “orbital” and “spin” parts,

$$\mathbf{J} = \int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV = \mathbf{L}+\mathbf{S}. \tag{9}$$

For this purpose they substitute [5]

$$\mathbf{B} = \nabla\times\mathbf{A}, \tag{10}$$

or [1]

$$\mathbf{E} = \nabla\times\mathbf{F}, \qquad \mathbf{F} = -\int\frac{\partial_t \mathbf{B}dV}{4\pi r}, \tag{11}$$

into (8). As a result, Eq. (10) gives

$$\mathbf{J} = \int \mathbf{r}\times(E^i\nabla A_i)dV + \int(\mathbf{E}\times\mathbf{A})dV , \tag{12}$$

and Eq. (11) gives

$$\mathbf{J} = \int \mathbf{r}\times(B^i\nabla F_i)dV + \int(\mathbf{F}\times\mathbf{B})dV . \tag{13}$$

for the beam. But I think these decompositions do not give grounds to interpret the summands as orbital and spin components of the angular momentum of the beam. Firstly, neither $\mathbf{E}\times\mathbf{A}$ nor $\mathbf{F}\times\mathbf{B}$ are electrodynamics spin tensors because the standard electrodynamics has no spin tensor. Secondly, the transformation of the integral over skin of the beam into an integral over bulk of the beam proves nothing. For example, consider $\int \mathbf{r}\times\mathbf{j}dV$ where **j** is an electric current density of a long solenoid. We have

$$\int \mathbf{r}\times\mathbf{j}dV = \int \mathbf{r}\times(\nabla\times\mathbf{H})dV$$
$$= \int(r^i\partial_k H_i - r^i\partial_i H_k)dV = \int[\partial_k(r^iH_i) - \partial_k r^i H_i - \partial_i(r^iH_k) + \partial_i r^i H_k]dV = \int 2\mathbf{H}dV$$

The equality between the integral of the moment of the current density over the surface of the solenoid and an integral of **H** over the interior of the solenoid proves nothing.

## 2. The truth

It was explained [6-16] that when modifying the canonical pair (1), (2) we must use another addends. Our addends are

$$t^{\lambda\mu} = \partial_\alpha A^\lambda F^{\mu\alpha} \quad \text{and} \quad s^{\lambda\mu\nu} = 2A^{[\lambda}\partial^{\mu]}A^\nu, \quad \partial_\lambda A^\lambda = 0, \tag{14}$$

instead of standard ones

$$\underset{st}{t}^{\lambda\mu} = -\partial_\nu(\underset{c}{\Upsilon}^{\lambda\mu\nu} - \underset{c}{\Upsilon}^{\mu\nu\lambda} + \underset{c}{\Upsilon}^{\nu\lambda\mu})/2, \quad \underset{st}{s}^{\lambda\mu\nu} = -\underset{c}{\Upsilon}^{\lambda\mu\nu}.$$

Our addends satisfy the equation

$$2t^{[\lambda\mu]} = \partial_\nu s^{\lambda\mu\nu}. \tag{15}$$

Using these addends we get, instead of (4), (6), the Maxwell energy-momentum tensor

$$T^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} + t^{\lambda\mu} = -F^\lambda{}_\alpha F^{\mu\alpha} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta}/4. \tag{16}$$

and a tensor, which was named an electric part of the spin tensor

$$\underset{e}{\Upsilon}^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + s^{\lambda\mu\nu} = 2A^{[\lambda}\partial^{|\nu|}A^{\mu]}. \tag{17}$$

This result was submitted to “JETP Letters” on May 12, 1998.

The spin tensor of electromagnetic waves must depend symmetrically on the magnetic vector potential $A_\alpha$ and on an electric vector potential

$$\Pi_\alpha = \varepsilon_{\alpha\lambda\mu\nu}\Pi^{\lambda\mu\nu}, \quad \partial_\nu \Pi^{\lambda\mu\nu} = F^{\lambda\mu}. \tag{18}$$

So the spin tensor of electromagnetic waves has the form

$$\Upsilon^{\lambda\mu\nu} = \underset{e}{\Upsilon}^{\lambda\mu\nu} + \underset{m}{\Upsilon}^{\lambda\mu\nu} = A^{[\lambda}\partial^{|\nu|}A^{\mu]} + \Pi^{[\lambda}\partial^{|\nu|}\Pi^{\mu]}. \tag{19}$$

This result was submitted to “JETP” on Jan. 27, 1999,

The use of the spin tensor (19) is presented in [6 – 16] and at the web site http://www.mai.ru/science/ . Absorption of a circularly polarized beam is calculated there, and a radiation of a rotating electrical dipole is considered

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