Inner incompleteness of the Maxwell electrodynamics

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A classical electrodynamics' spin tensor is introduced. So, the electrodynamics' ponderomotive action comprises a force from the Maxwell stress tensor and a torque from the spin tensor. A circularly polarized light beam without an azimuthal phase structure is considered. The beam carries a spin angular momentum and an orbital angular momentum. So, we double the beam's angular momentum in comparison with the Maxwell's theory result. Absorption of the beam by a target divided into two concentric parts is described.
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1. Circularly polarized beam

Accordingly to the traditional view, a total angular momentum of electromagnetic field is a moment of a linear momentum, and it is given by the formula [1 – 5]

$$\mathbf{J} = \int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV \,. \qquad (1)$$

In the case of a circularly polarized beam with a plane wavefront the angular momentum is localized on the surface of the beam because a falloff in intensity of the field on the surface gives rise E and B fields which are parallel to wave vector, and so, the mass-energy flow $\mathbf{E}\times\mathbf{H}$ has components, which are perpendicular to the wave vector [6 – 9]. So,

$$d\mathbf{J} = \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV$$

is directed in the propagation direction.

In my view, this angular momentum (1) is obviously an orbital angular momentum,

$$\mathbf{L} = \int \mathbf{r}\times(\mathbf{E}\times\mathbf{H})dV \,,$$

but they name it *spin* [8]: “The circulating energy flow in the wave implies the existence of angular momentum, whose direction is along the direction of propagation. This angular momentum is the spin of the wave”.

Within an inner region of the beam E and H fields are perpendicular to the wave vector, and the mass-energy flow is parallel to the wave vector. So, there is no angular momentum in the inner region.

The angular momentum of the beam (1) is calculated, for example, by Ohanian [8]

$$L = \int E_0^2 dV/\omega,$$

the energy of the beam is

$$\mathsf{E} = \int E_0^2 dV$$

where E_0 is the electric field in the inner region, and the ratio $\mathsf{E}/L = \omega$ is the same as the ratio $\hbar\omega/\hbar$, i.e. energy/spin, for a photon. That is why they name the angular momentum spin. Meanwhile, an important question was raised at the V. L. Ginsburg Moscow Physical Seminar in the spring of 1999. The question was about absorption of circularly polarized light by a round flat target, which is divided concentrically into an inner disc and a closely fitting outer annulus [10].

If the target absorbs a circularly polarized beam, the annulus absorbs the surface of the beam, which carries the angular momentum, and the disc absorbs the body of the beam, which has no angular momentum.

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According to the Maxwell theory, the expression $d\mathrm{F}^{j} = T^{ji} da_i$ gives an infinitesimal force acting on a surface element da_i (here T^{ji} is the Maxwell stress tensor, i.e. a space part of the Maxwell-Minkowski energy-momentum tensor). So, the disc does not perceive a torque when the target absorbs a circularly polarized beam,
since the Poynting vector is perpendicular to the disc, and there are no ponderomotive forces, which are capable to twist the disc. Tangential forces act only on the annulus.

But it is clear that in reality the disc does perceive a torque from the wave, since the disc gets spin, which is not described by the Maxwell theory. Beth [11], Feynman [12] clearly showed how circularly polarized wave transferred torque to medium. The disc will be twisted in contradiction with the paradigm.

Allen and Padgett [13] have attempted to explain the torque acting on the disc within the scope of the standard electrodynamics. They have written, “Any form of aperture introduces an intensity gradient,… and a field component is induced in the propagation direction and so the dilemma is potentially resolved”.' They decompose a plane wave into three beams: the inner beam, the annulus beam, and the remainder.

Alas! A small clearance between the inner disc and outer annulus does not aperture a beam and does not induce longitudinal field components. The imaginary decomposition of the plane wave is not capable to create longitudinal field components and, correspondingly, transverse momentum and a torque acting on the disc. Maxwell stress tensor cannot supply the disc with a torque. According to the Maxwell theory, the disc absorbs energy and feels pressure only.

To resolve the dilemma, in my opinion, we must use the conception of classical electrodynamics' spin which is described by a spin tensor $\Upsilon^{\mu\nu\alpha}$ [14 – 16]. So, we must recognize that the standard classical electrodynamics is not complete. Electrodynamics' spin tensor is not zero, and ponderomotive forces acting on a surface element da_i consist of both, the force itself and a torque:

$$d\mathrm{F}^{j} = T^{ji} da_i, \quad d\tau^{jk} = \Upsilon^{jki} da_i. \tag{2}$$

So, the annulus of our target perceives the orbital angular momentum $L = \mathrm{E}/\omega$ the disc perceives a spin angular momentum $S = \mathrm{E}/\omega$, and the target as a whole perceives a total angular momentum

$$J = L + S = 2\,\mathrm{E}/\omega .$$

So, we double the beam's angular momentum.

2. Classical electrodynamics' spin

Substance, which absorbs a circularly polarized electromagnetic wave, perceives, besides energy and momentum, spin angular momentum S. Its flux current density, Υ, is equal to $\Upsilon = \mathrm{S}/\omega$ where S is the modulus of the Poynting vector. This quantity is presented, for example, in [12]. But an expression of this quantity as a tensor density does not known, since the spin density is recognized as a zero in the modern electrodynamics, in contrast to an energy-momentum density, which equals the Maxwell-Minkowski tensor (density)

$$T^{\mu\alpha} = -g^{\mu\nu} F_{\nu\sigma} F^{\alpha\sigma} + g^{\mu\alpha} F_{\nu\sigma} F^{\nu\sigma}/4, \quad F_{\nu\sigma} = 2\partial_{[\nu} A_{\sigma]}$$

The Maxwell-Minkowski tensor is symmetric,

$$T^{\mu\alpha} = T^{(\mu\alpha)}, \quad T^{[\mu\alpha]} = 0$$

The divergence of the Maxwell-Minkowski tensor has the form

$$\partial_\alpha T^{\mu\alpha} = -j^{\sigma} F_{\nu\sigma} g^{\mu\nu}, \quad j^{\sigma} = -\partial_\nu F^{\sigma\nu}$$

As is known, in the case of gauge invariant Lagrangian

$$\mathrm{L}_c = -F_{\mu\nu} F^{\mu\nu}/4$$

standard Lagrange formalism gives the canonical energy-momentum and spin tensors [17]

$$\underset{c}{T}^{\mu\alpha} = -\partial^{\mu} A_{\sigma} F^{\alpha\sigma} + g^{\mu\alpha} F_{\nu\sigma} F^{\nu\sigma} / 4 , \quad \underset{c}{\Upsilon}^{\mu\nu\alpha} = -2A^{[\mu} F^{\nu]\alpha}$$

The canonical tensors are out of all relation to the physical reality [15,16,18]. The canonical energy-momentum tensor is nonsymmetric and has a wrong divergence

$$\underset{c}{T}^{[\mu\alpha]} = -\partial_{\sigma} (A^{[\mu} \partial^{\alpha]} A^{\sigma}) , \quad \partial_{\alpha} \underset{c}{T}^{\mu\alpha} = -j^{\sigma} \partial^{\mu} A_{\sigma}$$

To turn the canonical energy-momentum tensor to the Maxwell-Minkowski tensor theorists simply add a *ad hoc* term [18]

$$\underset{c}{T}^{\mu\alpha} + \partial_{\sigma} A^{\mu} F^{\alpha\sigma} = T^{\mu\alpha}$$

One might wonder what we must add to the canonical spin tensor for its repairing. Our answer is as follows: a spin addition, $s^{\mu\nu\alpha}$ and the energy-momentum addition,

$$t^{\mu\alpha} = \partial_{\sigma} A^{\mu} F^{\alpha\sigma}$$

must satisfy the equation

$$\partial_{\alpha} s^{\mu\nu\alpha} = 2t^{[\mu\nu]} \qquad (3)$$

It is easy to find from (3) that

$$s^{\mu\nu\alpha} = 2A^{[\mu} \partial^{\nu]} A^{\alpha}$$

and we obtain a variant of the electrodynamics spin tensor

$$\underset{e}{\Upsilon}^{\mu\nu\alpha} = \underset{c}{\Upsilon}^{\mu\nu\alpha} + s^{\mu\nu\alpha} = 2A^{[\mu} \partial^{|\alpha|} A^{\nu]} .$$

So, our spin tensor is a function of vector potential A^{μ} and is not gauge invariant. We greet this fact. As is shown [16], A^{μ} must satisfy the Lorentz condition, $\partial_{\mu} A^{\mu} = 0$.

Theorists adds another addends to the canonical tensors. The standard addends are

$$\underset{st}{t}^{\mu\alpha} = \partial_{\nu} (A^{\mu} F^{\alpha\nu}), \quad \underset{st}{s}^{\mu\nu\alpha} = 2A^{[\mu} F^{\nu]\alpha} = -\underset{c}{\Upsilon}^{\mu\nu\alpha} .$$

That is why a classical spin is absent in the modern electrodynamics. That is why they consider that a circularly polarized plane wave has no angular momentum.

3. Conclusion

Electrodynamics' spin tensor is not zero, and ponderomotive forces acting on a surface element consist of both, the force itself and a torque (2) in contradiction with the Maxwell theory. So, we must recognize that the standard classical electrodynamics is not complete. As an example, we double the total angular momentum of a circularly polarized beam with a plane wavefront in comparison with the Maxwell's theory.

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I am deeply grateful to Professor Robert H. Romer for publishing my question [10].

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Supplement

This paper was submitted to Physics Letters A on 22 Jul 2002. P.R. **Holland,** Editor, rejected it. Referee's report was:

This paper is intended to modify standard classical electrodynamics, since it is considered incomplete. The argument for that seems to be that spin would be missing from the general expression (1) for the angular momentum of an electromagnetic field. The argument is backed up by the example of a circularly polarized light beam, for which no component of (1) in the propagation direction remains, except at the edge, where the intensity decreases.
In my opinion, the argument is based on a number of miisunderstandings. In contrast to what the author states, the expression (1) is the complete angular momentum, not just orbital angular momentum (as he claims), and certainly not just spin (as he writes 'they name it spin': who are 'they'?).
In some cases J can be separated in a spin and an orbital part. This separation is not without problems, in view of the fact that the Maxwell field is transverse, and the tranversality is not maintained when only the polarization is rotated. However, for a light beam (with a given propagation direction) the angular momentum component along the beam can be separated in a spin and an orbital part (see e.g. Allen et al, PRA45, (1992) 8185, Van Enk and Nienhuis, Opt.Commun. 94 (1992) 147; Europhys. Lett. 25 (1994) 497). In the paraxial approximation, the orbital part is determined by the transverse momentum density, which is proportional to the transverse phase gradient of the field. The spin part of the angular momentum is proportional to the transverse intensity gradient. This latter fact is probably the basis of the author's statement that the angular momentum of a circularly polarized beam is localized on the surface of the beam. This poses no problem whatsoever, and absorption from the center of the beam transfers precisely one unit of \hbar per photon, as follows when one considers the difference in total angular momentum between the incoming and the transmitted beam. The statement that 'the disc does not perceive a torque when the target absorbs a circularly polarized beam' is wrong.
I see no justification for the strong claim that classical electrodynamics is incomplete, and the doubling of the beam's angular momentum mentioned at the end of p. 2 (without regard of the possible helicity of the phase distribution of the beam) is ununderstandable to me. In my view the author is tempting to remedy a problem that arises from a misconception of the concept of angular momentum of the Maxwell field. I advise rejection of the paper.

My answer was:

Dr **Holland**,
The referee asks, who names it spin? But there is the reference [8]. H. Ohanian, for example, names: "This angular momentum is the spin of the wave"
The referee claims that the orbital part is determined by the transverse momentum density, which is proportional to the transverse phase gradient of the field. It is correct. But in our case, the phase gradient is zero! And, so, Allen's orbital part is zero.
The referee claims that the spin part of the angular momentum is proportional to the transverse intensity gradient. But this is the very thing that "they" (Ohanian) name spin! And this angular momentum is localized on

the surface of the beam. But it is not spin. It is the orbital angular momentum. Spin is absent. Spin is absent in the Maxwell theory.

The referee claims that the center of the beam transfers precisely one unit of \hbar per photon. Ah! Maxwell theory does not know what is photon and what is \hbar! Maxwell theory knows stress tensor. According to the Maxwell theory, the center of the beam does not transfer angular momentum. It's trivial.

Yours sincerely, Radi Khrapko