# Absorption of a circularly polarized light beam

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**Abstract**

Absorption of a circularly polarized light beam without an azimuthal phase structure in a dielectric is considered in the frame of the classical electrodynamics. We calculate transferring of angular momentum and energy to the dielectric. Our result differs from the result of Loudon [1] (Phys. Rev. A**68**, 013806). It is found that the torque acting on the dielectric divides into surface and bulk parts in another way, and the total torque is twice as much as in the paper [1].

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## 1 Introduction. The Loudon's result

In the paper [1] expressions for an electromagnetic beam (L.2.3), (L.2.17), (L.2.18) are considered by the use of the quantum operators. The expressions get the form

$$u(x,y,z) \propto \exp\left[-\frac{x^2+y^2}{w_0^2} + \frac{iz(x^2+y^2)}{\breve{k}w_0^4} - \frac{2iz}{\breve{k}w_0^2}\right], \tag{1}$$

$$\breve{\mathbf{E}} = \exp[i(\breve{k}z - t)][\mathbf{x} + i\mathbf{y} + \frac{\mathbf{z}}{\breve{k}}(i\partial_x - \partial_y)]u, \quad \breve{\mathbf{B}} = -i\breve{k}\breve{\mathbf{E}} \tag{2}$$

when the beam has no azimuth phase structure ($l = 0$) and the beam polarization is circular ($\alpha = -i/\sqrt{2},\ \beta = 1/\sqrt{2},\ \sigma = 1$). Notations like (L.2.3) mean references to the corresponding formulas of [1]. $w_0$ is the (constant) beam waist. For short, we set speed of light in vacuum, $c = 1$, and frequency, $\omega = 1$. The symbol 'breve' marks complex vectors and numbers. $\breve{k} = \eta + i\kappa$ is the complex wave number (L.6.1).

If a beam of the type (2) with $\breve{k} = 1$ for $z < 0$,

$$\breve{\mathbf{E}}_1 = \exp[i(z - t)][\mathbf{x} + i\mathbf{y} + \mathbf{z}(i\partial_x - \partial_y)]u, \quad \breve{\mathbf{B}}_1 = -i\breve{\mathbf{E}}_1, \tag{3}$$

impinges normally on a surface of a dielectric which is characterized by $\breve{k}$, the beam divides into a reflected part (for $z < 0$)

$$\breve{\mathbf{E}}_2 = \frac{1-\breve{k}}{1+\breve{k}}\exp[i(-z - t)][\mathbf{x} + i\mathbf{y} - \mathbf{z}(i\partial_x - \partial_y)]u, \quad \breve{\mathbf{B}}_2 = i\breve{\mathbf{E}}_2 \tag{4}$$

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and a trasmitted part (for $z > 0$)

$$\breve{\mathbf{E}}_3 = \frac{2}{1+\breve{k}} \exp[i(\breve{k}z - t)][\mathbf{x} + i\mathbf{y} + \frac{\mathbf{z}}{\breve{k}}(i\partial_x - \partial_y)]u, \quad \breve{\mathbf{B}}_3 = -i\breve{k}\breve{\mathbf{E}}_3 \tag{5}$$

in accordance with the reflected (L.6.6) and the transmission (L.6.9) coefficients

$$\breve{R} = \frac{1-\breve{k}}{1+\breve{k}}, \quad \breve{T} = \frac{2}{1+\breve{k}}.$$

If we set

$$\int | u |^2 \, dxdy = 1, \tag{6}$$

the average power that enters the dielectric, according to [1], is (L.6.10), (L.6.14) (cf. (22))

$$\mathcal{P} = \eta \mid \breve{T} \mid^2 = \frac{4\eta}{(1+\eta)^2 + \kappa^2}. \tag{7}$$

Loudon states that the average total transfer of angular momentum to the dielectric per unit time, i.e. torque, is (L.7.2),

$$\tau = \frac{4\eta}{(1+\eta)^2 + \kappa^2}, \tag{8}$$

and the torque consists of surface and bulk parts (L.7.24):

$$\tau = \frac{4\eta}{(1+\eta)^2 + \kappa^2} = \overbrace{\frac{4\eta(k^2-1)}{[(1+\eta)^2 + \kappa^2]k^2}}^{surface} + \overbrace{\frac{4\eta}{[(1+\eta)^2 + \kappa^2]k^2}}^{bulk}, \quad (k^2 = \mid \breve{k} \mid^2). \tag{9}$$

But Loudon ignores $\mathbf{E} \times \mathbf{P}$ term in (L.7.18).

We present a recalculation of the power and the torque by the use of the classical electrodynamics (see also [2]), but firstly, we must note that because of the paraxial assumption, $\mid \partial_z u \mid << \mid \breve{k} \mid$ (L.2.7), or $\mid \breve{k} \mid >> 1/w_0$ (L.2.8), two last terms in (1) may be neglected in compare with $i\breve{k}z$. So, we may consider

$$u(x, y) \propto \exp\left(-\frac{x^2 + y^2}{w_0^2}\right)$$

as $u(x, y, z)$. Then the beam (2) is equivalent to the Jackson's beam [3, p. 201]. Jackson considers a circularly polarized plane wave moving in the $z$ direction, which has a finite extend in the x and y directions:

$$\breve{\mathbf{E}}(x, y, z) = \exp[i(\breve{k}z - t)][\mathbf{x} + i\mathbf{y} + \frac{\mathbf{z}}{\breve{k}}(i\partial_x - \partial_y)]E_0(x, y), \quad \breve{\mathbf{B}} = -i\breve{k}\breve{\mathbf{E}} \tag{10}$$

# 2 Cylindrical coordinates

We use cylindrical coordinates $\rho, \phi, z$,

$$x = \rho \cos\phi, \quad y = \rho \sin\phi,$$

with the metric

$$dl^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2, \quad g_{\rho\rho} = 1, \quad g_{\phi\phi} = \rho^2, \quad g_{zz} = 1, \quad \sqrt{g}_\wedge = \rho, \quad g^{\phi\phi} = 1/\rho^2.$$

Square root of determinant of the metric tensor is a scalar density of weight +1. It is marked by a symbol 'wedge' at a level of the bottom indexes. Volume element is a density of weight −1 and is marked by 'wedge' at a level of the upper indexes, $dV^\wedge = d\rho d\phi dz$, as well as the absolut antisymmetric density $e^\wedge_{ijk}$, which equals ±1, or 0.

A transformation of *covariant* components of the vectors **E**, **B** in (3), (4), (5), for example

$$E_\phi = \partial^x_\phi E_x + \partial^y_\phi E_y + \partial^z_\phi E_z = (-\rho\sin\phi + i\rho\cos\phi)\exp[i(z-t)]u = i\exp[i(z-t+\phi)]\rho u(\rho),$$

$$E_z = \exp[i(z-t)](i\partial_x - \partial_y)u = \exp[i(z-t)](i\cos\phi - \sin\phi)\partial_\rho u = i\exp[i(z-t+\phi)]\partial_\rho u,$$

gives

$$\underset{\to}{\breve{E}}_1 = \exp[i(z-t+\phi)](\underset{\to}{\rho} + i\rho\underset{\to}{\phi} + \underset{\to}{z}\, i\partial_\rho)u(\rho), \quad \underset{\to}{\breve{B}}_1 = -i\underset{\to}{\breve{E}}_1, \tag{11}$$

$$\underset{\to}{\breve{E}}_2 = \frac{1-\breve{k}}{1+\breve{k}}\exp[i(-z-t+\phi)](\underset{\to}{\rho} + i\rho\underset{\to}{\phi} - \underset{\to}{z}\, i\partial_\rho)u(\rho), \quad \underset{\to}{\breve{B}}_2 = i\underset{\to}{\breve{E}}_2, \tag{12}$$

$$\underset{\to}{\breve{E}}_3 = \frac{2}{1+\breve{k}}\exp[i(\breve{k}z-t+\phi)](\underset{\to}{\rho} + i\rho\underset{\to}{\phi} + \underset{\to}{z}\frac{i}{\breve{k}}\partial_\rho)u(\rho), \quad \underset{\to}{\breve{B}}_3 = -i\breve{k}\underset{\to}{\breve{E}}_3, \tag{13}$$

The arrow placed under a symbol means a covariant vector, or a covariant coordinate vector.

# 3 Dielectric

When an electromagnetic wave passes through a dielectric, the electric field polarizes the dielectric. The electric polarization of the dielectric, time derivative of it, i.e. the displacement current, and the Lorentz force density acting on the current have the forms

$$\mathbf{P} = (\breve{\epsilon} - 1)\mathbf{E}, \quad \mathbf{j} = \partial_t \mathbf{P}, \quad \mathbf{f} = \mathbf{j}\times\mathbf{B}, \quad \breve{\epsilon} = \breve{k}^2. \tag{14}$$

Besides that, the circular polarization of the electromagnetic wave gives rise to a torque. The torque per unit volume produced by the action of the electric field on the polarization of the dielectric is [4]

$$\mathbf{I} = \mathbf{P}\times\mathbf{E}. \tag{15}$$

We consider firstly z-component of the vector prouct $(\mathbf{r}\times\mathbf{f})_z$ from (14). We interpret the torque $\underset{f}{\tau}{}_z$ produced by the force **f** as a bulk contribution. The torque equals the integral of

$$d\underset{f}{\tau}{}_z = \rho f_{z\rho}\sqrt{g}_\wedge dV^\wedge \tag{16}$$

over the dielectric ($z > 0$).

To calculate the integral, we must substitute

$$\breve{E}_{3\rho} = \frac{2}{1+\breve{k}}\exp[i(\breve{k}z-t+\phi)]u(\rho), \quad \breve{E}_{3z} = \frac{2i}{(1+\breve{k})\breve{k}}\exp[i(\breve{k}z-t+\phi)]\partial_\rho u(\rho),$$

$$\breve{B}_{3\rho} = -i\breve{k}\breve{E}_{3\rho}, \quad \breve{B}_{3z} = -i\breve{k}\breve{E}_{3z}$$

from (13) into (16). Integrating with respect to $\phi, z$ and time averaging yields

$$\underset{f}{\tau}{}_z = \pi \int \rho^2 \Re[(\breve{\epsilon}-1)(\partial_t \breve{E}_{3z}\overline{B}_{3\rho} - \partial_t \breve{E}_{3\rho}\overline{B}_{3z})] d\rho dz$$
$$= \frac{2\pi}{\kappa[(1+\eta)^2+\kappa^2]} \int \rho^2 \Re\left[ i(\breve{\epsilon}-1)\left(\frac{\overline{k}}{\breve{\overline{k}}}+1\right)\right] \partial_\rho (u^2/2) d\rho.$$

The over lines mark complex conjugate complex numbers. Due to (6), integrating by parts yields the time average bulk torque acting on the dielectric

$$\underset{f}{\tau}{}_z = -\Re\left[ i(\breve{\epsilon}-1)\left(\frac{\overline{k}}{\breve{\overline{k}}}+1\right)\right] \frac{1}{\kappa[(1+\eta)^2+\kappa^2]} = \frac{2\eta(k^2+1)}{k^2[(1+\eta)^2+\kappa^2]}. \tag{17}$$

We interpret this torque as a bulk part of orbital angular momentum absorbed per unit time.

Now we calculate an integral of density (15) using (13),

$$\tau^z = \int \Re(\breve{P}_\rho \overline{E}_{3\phi} - \breve{P}_\phi \overline{E}_{3\rho}) e^{\rho\phi z}_{\wedge} (d\rho d\phi dz)^{\wedge}/2 = \frac{4\eta}{(1+\eta)^2+\kappa^2}. \tag{18}$$

We interpret this torque as spin absorbed per unit time. This torque is absent in [1].

# 4 Space in front of the dielectric. Surface torque. Total torque and power

At the boundary of the dielectric, $z=0$, according to (3), (4), (5), all components of $\mathbf{B}$ and tangent components of $\mathbf{E}$ are continuous, but $E_z$ reduces by a factor of $\breve{\epsilon}$. It means a presence of electrical charge at the boundary. Due to (11), (12), (13) the density of the charge is

$$\breve{\sigma} = [\breve{E}_{3z} - \breve{E}_{1z} - \breve{E}_{2z}]_{z=0} = \frac{2i(1-\breve{k}^2)}{(1+\breve{k})\breve{k}} \exp[i(-t+\phi)]\partial_\rho u(\rho).$$

The charge experiences tangential forces $\sigma E_\phi$ from the electrical field; moment of the forces is a surface torque acting on our dielectric. Integrating by parts and time averaging yields

$$\underset{\sigma}{\tau} = \int \rho \Re(\breve{\sigma}\overline{E}_{3\phi}) d\rho d\phi/2 = \frac{4\pi}{(1+\eta)^2+\kappa^2} \int \rho^2 \partial_\rho (u^2/2) \Re\left[\frac{1-\breve{k}^2}{\breve{k}}\right] d\rho = \frac{2\eta(k^2-1)}{k^2[(1+\eta)^2+\kappa^2]}. \tag{19}$$

We interpret this torque as a surface part of the orbital angular momentum absorbed per unit time. It is remarkable that the sum of the surface part of torque (19) and the bulk part of torque (17) equals the spin part of torque (18).

$$\underset{\sigma}{\tau} + \underset{f}{\tau} = \overset{surface\ orbital}{\frac{2\eta(k^2-1)}{k^2[(1+\eta)^2+\kappa^2]}} + \overset{bulk\ orbital}{\frac{2\eta(k^2+1)}{k^2[(1+\eta)^2+\kappa^2]}} = \overset{bulk\ spin}{\frac{4\eta}{(1+\eta)^2+\kappa^2}} = \tau. \tag{20}$$

So, the total torque experienced by our dielectric equals the double quantity

$$\underset{tot}{\tau} = \underset{\sigma}{\tau} + \underset{f}{\tau} + \tau = \frac{8\eta}{(1+\eta)^2+\kappa^2} \tag{21}$$

One can calculate the power transmitting from the beam to the dielectric by integrating the Poynting vector over a section of the beam and by time averaging (cf. (7))

$$\mathcal{P} = \int \Re[(\breve{E}_{1\rho} + \breve{E}_{2\rho})(\overline{B}_{1\phi} + \overline{B}_{2\phi}) - (\breve{E}_{1\phi} + \breve{E}_{2\phi})(\overline{B}_{1\rho} + \overline{B}_{2\rho})]d\phi d\rho/2 = \frac{4\eta}{(1+\eta)^2 + \kappa^2} \quad (22).$$

So, our result is

$$\underset{tot}{\tau}/\mathcal{P} = 2 \quad (\omega = 1).$$

One can find an explanation of the difference between our results and the results of [1] in [2, 5, 6, 7, 8]

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# References


[1] R. Loudon, Phys. Rev. A**68** 013806 (2003).

[2] R. I. Khrapko,
http://www.mai.ru/projects/mai_works/articles/num14/article6/auther.htm (2003).

[3] J. D. Jackson, *Classical Electrodynamics,* (John Wiley, New York, 1962), p. 201.

[4] R. A. Beth, Phys. Rev. **48**, 471 (1935); Phys. Rev. **50**, 115 (1936).

[5] R. I. Khrapko, physics/0102084, physics/0105031.

[6] R. I. Khrapko, Measurement Techniques **46**, No.4, 317 (2003).

[7] R. I. Khrapko, mp_arc 03-307, mp_arc 03-311.

[8] R. I. Khrapko, mp_arc 03-315.

[9] R. I. Khrapko, Amer. J. Phys. **69**, 405, (2001).