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Corresponding Author: Radi Khrapko,

Corresponding Author's Institution: Moscow aviation institute

First Author: Radi Khrapko

Order of Authors: Radi Khrapko

Highlights

- The idea that electrodynamics' spin is a moment of momentum seems to be incorrect.
- Spin and moment of momentum are different concepts.
- Spin is a half of the moment of momentum for a rotating dipole radiation.
- Thus Jackson and Becker are mistaken.
- Besides, these concepts are spatially separated.

*Manuscript

Spin is not a moment of momentum

Radi I. Khrapko[*]
Moscow Aviation Institute, 125993, Moscow, Russia

The modern idea that spin of an electromagnetic radiation is a moment of a linear momentum seems to be incorrect. The idea is founded on the equality between the moment of momentum and the integral of a component of the canonical spin tensor for a light beam. We show that Jackson and Becker generalized this equality into a radiation of a localized source erroneously. In reality, the integral of the spin tensor is a half of the moment of momentum for the case of a rotating electric dipole (because photons are variously directed, and their spins are not parallel to each other as in a beam). Besides, the moment of momentum and the spin, represented by the spin tensor, are spatially separated. Thus, moment of momentum and spin are different concepts. We use the spin tensor and calculate spin separately from moment of momentum.

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1. Introduction

According to the modern electrodynamics, spin of an electromagnetic radiation is a moment of linear momentum of this radiation [1] or a part of the moment. Usually this statement is illustrated by the use of a circularly polarized light beam [2] or by a radiation of source localized to a finite region of space [3, problem 7.27], [4, p. 320]. This statement is substantiated by so called Humblet transformation [5], which transforms the moment of momentum $\mathbf{J}$ of a section of such a beam into an integral of the quantity $\mathbf{E}\times\mathbf{A}$:

$$\mathbf{J} = \varepsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV = \varepsilon_0\int E^i(\mathbf{r}\times\nabla)A_i dV + \varepsilon_0\int \mathbf{E}\times\mathbf{A}dV \,. \quad (1.1)$$

The quantity $\varepsilon_0\mathbf{E}\times\mathbf{A}$ is interpreted as a density of electromagnetic spin.

This transformation implies integration by parts and is presented as a decomposition of the moment of momentum $\mathbf{J}$ into orbital and spin parts. However, the first term on the right, which is posed as the orbital part, obviously is zero for a symmetric beam [6], e.g. for the Jackson's beam [3, problem 7.28]

$$\mathbf{E} = \exp(ikz - i\omega t)[\mathbf{x} + i\mathbf{y} + \frac{\mathbf{z}}{k}(i\partial_x - \partial_y)]E_0(x,y), \quad \mathbf{B} = -i\mathbf{E}/c \,. \quad (1.2)$$

This means the total moment of momentum in such a beam is spin.

However, there is a serious geometrical objection against the identification of the density of moment of momentum, $\varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$, with the spin density, $\varepsilon_0\mathbf{E}\times\mathbf{A}$, in spite of the equality of integrals (1.1). The point is $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ is localized on the surface of the beam. A flow of circulating mass-energy flows there. On the contrary, $\mathbf{E}\times\mathbf{A}$ is distributed over the beam's body. Therefore, in paper [6], a conclusion was made that integrating of quantity $\mathbf{E}\times\mathbf{A}$ is simply a method by which the moment of momentum of the circulating flow can be calculated and that the moment is an orbital angular momentum

$$\mathbf{L} = \varepsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV \,. \quad (1.3)$$

Nevertheless, Ohanian [2] claims, "this angular momentum is the spin of the wave".

Jackson [3] and Becker [4] are united with Ohanian and Heitler. To confirm the identity between moment of momentum and spin, Jackson [3, problem 7.27] and Becker [4, V. 2, p. 320] consider an electromagnetic radiation produced by a source localized in a finite region of space. They apply the Humblet transformation with the integration by parts to the radiation and obtain the same equality as (1.1)

$$\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV = \int \mathbf{E}\times\mathbf{A}dV \,. \quad (1.4)$$

[*] Email: khrapko_ri@hotmail.com.

But they are mistaken! The equality (1.1), (1.4) proves to be invalid in this case. Their derivation of formula (1.4) contains a mathematical mistake because the integration by parts cannot be used when radiating into space. A straight calculation presented in Section 5 for the radiation of a rotating dipole gives

$$\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV = 2\int \mathbf{E}\times\mathbf{A}dV . \tag{1.5}$$

Somewhat such result must be expected if we attribute the sense of spin density to the integrand $\mathbf{E}\times\mathbf{A}$ because when radiating into space photons are variously directed, and their spins are not parallel to each other as in a beam. But, the point is result (1.5) proves the moment of momentum, $\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV$, is not spin, $\int \mathbf{E}\times\mathbf{A}dV$!

Besides, as in the case of the beam, the quantities $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ and $\mathbf{E}\times\mathbf{A}$ are spatially separated in the case of the dipole radiation: moment of momentum, $\varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$, is radiated mainly near the plane of rotating of the dipole, while spin, $\varepsilon_0\mathbf{E}\times\mathbf{A}$, exists near the rotating axis, where the radiation is circularly or elliptically polarized [8].

There is one more important circumstance, which prevent the interpretation of the integral $\varepsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV$ as spin of a *radiation*. Vectors $\mathbf{E}$ and $\mathbf{B}$ of a radiation are perpendicular to the direction of the propagation i.e. $(\mathbf{E}\times\mathbf{B})\cdot\mathbf{k}=0$, where $\mathbf{k}$ is the wave vector. So $(\mathbf{r}\times(\mathbf{E}\times\mathbf{B}))\cdot\mathbf{k}=0$ for any radiation. Therefore the moment of momentum $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ must be calculated by the use of the non-radiative field, which is proportional to $1/r^2$ in the case of a radiation into space. This indicate non-radiative nature of the moment of momentum $\varepsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV$ while spin is an attribute of a radiation and must be calculated by the use of fields, which is proportional to $1/r$ only. Heitler, when defending spin nature of the moment of momentum, refers to a subtle interference effect on this subject [1]. But this explanation seems to be not convincing.

On our opinion, it is necessary to concede that $\varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ represents a moment of momentum, which has an orbital nature and does not represent spin of an electromagnetic radiation [6-9]. We use $\varepsilon_0\mathbf{E}\times\mathbf{A}$ as the spin density in Section 5.

2. Scheme of the calculations

There are two methods of calculations of energy, moment of momentum and spin fluxes of an electromagnetic radiation. These methods of course give identical results.

1. Volume density (of mass-energy or of moment of momentum) is integrated over a thin spherical layer (of thickness dr), which surrounds the source of the radiation, and then the integral is divided by dt, on the assumption $dr/dt = c$. So, the formulas for power of radiation and torque are obtained:

$$P = \int T^{tt} da_i dr^i / dt , \quad \tau^{ij} = \int 2r^{[i}T^{j]t} da_k dr^k / dt \ \text{[J]}. \tag{2.1}$$

Here components of the Maxwell energy-momentum tensor are used: T^{tt} is the volume density of mass-energy and T^{jt} is volume density of momentum, which is equal to the Poynting vector because of the symmetry of the Maxwell energy-momentum tensor,

$$T^{tt} = \varepsilon_0 E^2/2 + \mu_0 B^2/2 , \quad T^{jt} = T^{tj} = \varepsilon_0 \mathbf{E}\times\mathbf{B} \ \text{[kg/m}^2\text{s]}. \tag{2.2}$$

2. However, it is more natural to integrate flux density components of the Maxwell tensor over a surface, which surrounds the source of the radiation:

$$P = \int T^{ti} da_i \ \text{[kg/s]}, \quad \tau^{ij} = \int 2r^{[i}T^{j]k} da_k , \tag{2.3}$$

here T^{jk} is the Maxwell stress tensor.

The same two methods are applicable for a calculation of a spin radiation:

$$\underset{s}{\tau}^{ij} = \int \Upsilon^{ijt} da_k dr^k / dt \ \text{[J]}, \tag{2.4}$$

$$\underset{s}{\tau}^{ij} = \int \Upsilon^{ijk} da_k \ , \tag{2.5}$$

where $\Upsilon^{ijt} = \varepsilon_0 \mathbf{E} \times \mathbf{A}$, $\Upsilon^{\lambda\mu\nu}$ is a spin tensor.

We show by a straight calculation that, in the field of a rotating dipole, the ratio of power to moment of momentum flux,

$$P / \tau = \omega , \tag{2.6}$$

differs from the ratio of power to spin flux,

$$P / \underset{s}{\tau} = 2\omega , \tag{2.7}$$

in accordance with formula (1.5), and thus a moment of momentum is not spin. It is important that the ratio $P / \underset{s}{\tau}$ is the normal ratio of energy to spin for circularly polarized photons directed along the rotation axis $(\theta = 0)$,

$$\left\lfloor P / \underset{s}{\tau} \right\rfloor_{\theta=0} = \hbar\omega / \hbar = \omega , \tag{2.8}$$

rather than ratio (2.7).

We use the complex expressions for electromagnetic fields [4, V.1, p.284], [10, p.36],

$$\mathbf{E} = \left[\frac{\omega^2 (\mathbf{p} r^2 - (\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0 c^2 r^3} + \frac{i\omega(\mathbf{p} r^2 - 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 c r^4} - \frac{(\mathbf{p} r^2 - 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 r^5} \right] \exp(ikz - i\omega t) \tag{2.9}$$

$$\mathbf{H} = \left[\frac{\omega^2 \mathbf{r} \times \mathbf{p}}{4\pi c r^2} + \frac{i\omega \mathbf{r} \times \mathbf{p}}{4\pi r^3} \right] \exp(ikz - i\omega t) \tag{2.10}$$

The calculation of the power P by the method (2.3) is performed in [10, p.39] with a mistake. We give this calculation in Section 3, having corrected the mistake. The calculation of the moment of momentum flux τ^{xy} by the method (2.1) is performed in [10, p.41] with a mistake as well. We give this calculation in Section 4, having corrected the mistake. The calculation of the spin flux $\underset{s}{\tau}^{ij}$ by the method (2.4) is performed in Section 5.

3. Calculation of radiation power by method (2.3)

We integrate the Poynting vector T^{ti} (2.2) over a spherical surface of radius r :

$$P = \int T^{ti} da_i = \Re \int \varepsilon_0 \mu_0 (\mathbf{E} \times \overline{\mathbf{H}}) \mathbf{r} r d\Omega / 2 = \Re \int \mathbf{E} (\overline{\mathbf{H}} \times \mathbf{r}) r d\Omega / 2c^2 \ , \tag{3.1}$$

$d\Omega = \sin\theta d\theta d\varphi$, the line means complex conjugation. Substituting fields proportional to $1/r$ from (2.9), (2.10) yields

$$P = \int \frac{\omega^4 \, | \mathbf{p} r^2 - (\mathbf{pr})\mathbf{r} |^2}{32\pi^2 c^5 \varepsilon_0 r^4} d\Omega . \tag{3.2}$$

This expression coincides with formula (2.71) in [10]. Using Cartesian components of the single dipole rotating in x-y plain yields

$$p_x = \exp(-i\omega t), \ p_y = i \exp(-i\omega t) \ \text{[C m]}. \tag{3.3}$$

We obtain

$$\begin{aligned} &[\mathbf{p} r^2 - (\mathbf{pr})\mathbf{r}] \cdot [\overline{\mathbf{p}} r^2 - (\overline{\mathbf{p}}\mathbf{r})\mathbf{r}] / r^4 = \mathbf{p}\overline{\mathbf{p}} - (\mathbf{pr})(\overline{\mathbf{p}}\mathbf{r}) / r^2 = \\ &= p_x \overline{p}_x + p_y \overline{p}_y - (x + iy)(x - iy) / r^2 = 2 - \sin^2\theta = 1 + \cos^2\theta. \end{aligned} \tag{3.4}$$

So,

$$P = \int \frac{\omega^4 (1 + \cos^2\theta) \sin\theta}{32\pi^2 c^5 \varepsilon_0} d\theta d\varphi . \tag{3.5}$$

This result was obtained also as a solution of problem 1 in [11, § 67], but formula (2.73) in [10] inexplicably gives half of the quantity:

$$dP = \frac{\omega^4 (1 + \cos^2\theta)}{64\pi^2 c^5 \varepsilon_0} d\Omega .$$

So, mass-energy flux in the field of a rotating dipole is

$$P = \frac{\omega^4}{6\pi c^5 \varepsilon_0} \text{ [kg/s]}. \quad (3.6)$$

This result is twice as much as the result [10, (2.74)].

4. Calculation of moment of momentum flux by method (2.1)

We integrate the moment of momentum volume density over a spherical layer

$$\tau^{ij} = \int 2r^{[i}T^{j]t}da_k dr^k / dt = \Re\int \varepsilon_0\mu_0 \mathbf{r}\times(\mathbf{E}\times\overline{\mathbf{H}})r^2 cd\Omega/2 = \Re\int [\mathbf{E}(\mathbf{r}\overline{\mathbf{H}}) - (\mathbf{rE})\overline{\mathbf{H}}]r^2 d\Omega/2c. \quad (4.1)$$

The first term on the right is zero, and the second term needs the use of the electromagnetic field, which is proportional to $1/r^2$

$$\tau^{ij} = \Re\int \mathbf{r}\frac{i\omega(-\mathbf{p}r^2 + 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 cr^4}\overline{\mathbf{H}}r^2 d\Omega/2c = \Re\int \frac{i\omega 2(\mathbf{rp})}{4\pi\varepsilon_0 cr^2}\frac{\omega^2 \mathbf{r}\times\overline{\mathbf{p}}}{4\pi c^2}d\Omega/2 = \Re\int \frac{i\omega^3(\mathbf{rp})\mathbf{r}\times\overline{\mathbf{p}}}{16\pi^2\varepsilon_0 c^3 r^2}d\Omega. \quad (4.2)$$

This expression coincides with formula (2.78) in [10]. Since (3.3) we obtain

$$[(\mathbf{rp})\mathbf{r}\times\overline{\mathbf{p}}]/r^2 = [(xp_x + yp_y)(x\overline{p}_y - y\overline{p}_x)]/r^2 = -i(x^2+y^2)/r^2 = -i\sin^2\theta. \quad (4.3)$$

As $\int_0^\pi \sin^3\theta d\theta = 4/3$, the torque emitted by the radiator is equal to

$$\tau^{xy} = \frac{\omega^3}{16\pi^2\varepsilon_0 c^3}\int \sin^3\theta d\theta d\varphi = \frac{\omega^3}{6\pi\varepsilon_0 c^3} \text{ [J]}. \quad (4.4)$$

Contrary to (4.4), formula (2.80) in [10] inexplicably gives half of quantity (4.4), $\tau_z = \omega^3/12\pi^2\varepsilon_0 c^3$. However, somehow the ratio of power to moment of momentum flux is equal to frequency (2.6)

$$c^2 P/\tau = \omega. \quad (4.5)$$

5. Calculation of spin flux by method (2.4)

We integrate the spin volume density, $\Upsilon^{ijt} = \varepsilon_0 \mathbf{E}\times\mathbf{A}$, over a spherical layer

$$\underset{s}{\tau}^{xy} = \Re\int \varepsilon_0 E_{[x}\overline{A}_{y]}r^2 d\Omega dr/dt = \Re\int i\varepsilon_0 E_{[x}\overline{E}_{y]}r^2 cd\Omega/\omega. \quad (5.1)$$

By the use $\mathbf{E}$ from (2.9), which is proportional to $1/r$, and since (3.3) we obtain

$$E_{[x}\overline{E}_{y]}r^2 = (E_x\overline{E}_y - E_y\overline{E}_x)r^2/2 =$$

$$= \frac{\omega^4}{32\pi^2\varepsilon_0^2 c^4 r^4}\{[r^2 - (x+iy)x][-ir^2 - (x-iy)y] - [ir^2 - (x+iy)y][r^2 - (x-iy)x]\} = \quad (5.2)$$

$$= \frac{-i\omega^4}{16\pi^2\varepsilon_0^2 c^4}(1 - \frac{x^2+y^2}{r^2}) = \frac{-i\omega^4}{16\pi^2\varepsilon_0^2 c^4}(1-\sin^2\theta) = \frac{-i\omega^4}{16\pi^2\varepsilon_0^2 c^4}\cos^2\theta.$$

As $\int_0^\pi \cos^2\theta\sin\theta d\theta = 2/3$, the spin flux emitted by the radiator is equal to

$$\underset{s}{\tau}^{xy} = \frac{\omega^3}{12\pi\varepsilon_0 c^3} \text{ [J]}. \quad (5.3)$$

This is half of the moment of momentum flux (4.4).

This result (5.3) was obtained by method (2.5) and by the use of spherical coordinates in paper [8]. In that paper, also the Corney's mistakes were indicated.

6. Conclusions, comments, and acknowledgements

A separate existence of spin and moment of momentum as different physical concepts is emphasized. These concepts originates in the Lagrange formalism with Noether's theorem where the canonical energy-momentum and spin tensor come into existence. We use the time component of the canonical spin tensor. We have ascertained equality (1.4) for the light beam is accidental.

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