Spin is not a moment of momentum

Radi I. Khrapko*
Moscow Aviation Institute, 125993, Moscow, Russia

Unambiguous definitions of energy-momentum and spin tensors are cited. Moment of momentum and spin are shown to be different concepts, but spin is absent in the modern electrodynamics. Nevertheless, moment of momentum and spin of a rotating dipole radiation is calculated, and notice is taken of principal mistakes of Jackson and Becker. This means that the equality between moment of momentum and canonical spin is false.

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A strange delusion is widespread that no unambiguous definition of the electrodynamics' energy-momentum tensor is possible. The paper "Energy-momentum localization and spin" [1] is ignored. Now we call attention to this curious paradox once more.

1. What is the energy-momentum tensor?

As an example of an unambiguous definition of the energy-momentum tensor we recall J. L. Synge's definitions [2]: "We assign to a material continuum a symmetric energy-momentum tensor. The tensor embodies the mechanical properties of the matter, such as stress and density". And we recall also that an electromagnetic field is a material continuum. Synge states the interpretation of the energy-momentum tensor in terms of flux densities, and he makes the following statement concerning the energy-momentum tensor $T^{\lambda\mu}$:

$$\text{4-momentum } dp^\lambda \text{ across a 3-target } dV_\mu \text{ is } dp^\lambda = T^{\lambda\mu} dV_\mu . \qquad (1.1)$$

If the 3-target is an infinitesimal 3-volume at rest relative to an observer's laboratory, then $dV_j = 0, \ (j = 1, 2, 3)$, and $dp^\lambda = T^{\lambda t} dV_t$ is the infinitesimal 4-momentum of the material continuum within the 3-volume, i.e. $dm = dp^t = T^{tt} dV_t$ is the mass within the 3-volume, and $dp^i = T^{it} dV_t$ is the 3-momentum within the 3-volume, i.e. T^{tt} and T^{it} are the mass and momentum density of the continuum, respectively.

If the 3-target is a surface element da_j, then $dV_t = 0, \ dV_j = da_j dt$, and $dp^\lambda = T^{\lambda j} da_j dt$, i.e.

$$dp^t / dt = T^{tj} da_j \ \text{ and } \ F^i = dp^i / dt = T^{ij} da_j \qquad (1.2)$$

are the mass-energy flux (power) across the surface element da_j and the force acting on the surface element da_j, respectively, i.e. T^{tj} is mass-energy flux density in the continuum, and T^{ij} is stress tensor of the continuum. If the material continuum is an electromagnetic field, T^{tj} is called the Poynting vector, and T^{ij} is called the Maxwell stress tensor.

At great length [3], "the component T^{ij} of the stress tensor is the i th component of the force on unit area perpendicular to the x_j-axis. For instance, the force on unit area perpendicular to the x-axis, normal to the area (i.e. along the x-axis), is T^{xx}, and the tangential forces on unit area (along the y and z axis) are T^{yx} and T^{zx}". In other words, T^{xx} is the pressue on a surface element da_x.

The local definition (1.1) of the energy-momentum tensor are valid not only for an electromagnetic field. If the material continuum is a solid body, the stress tensor depends on a deformation of the body. The deformation is described mathemanically by the strain tensor u^{ij} [3], and the stress tensor is determined by the form

$$T^{ij} = K u^l_l \delta^{ij} + 2\mu (u^{ij} - u^l_l \delta^{ij} / 3), \qquad (1.3)$$

* Email: khrapko_ri@hotmail.com.

where K and μ are moduluses of compression and shear, respectively [3]. We present this form here to emphasize that if the deformation is known, the stress tensor is determined unambiguously, and tension sensors can check the stress.

Analogically, if a pressure of light is measured, or Faraday's tensions along lines of force and pressures at right angles to lines of force are measured, the Maxwell stress tensor T^{ij} is determined unambiguously, according to the local definition (1.1). My radio receiver determines the Poynting vector T^{tj} locally and unambiguously (in a frequency interval). Alan Corney [4] demonstrates the angular dependence of the Poynting vector in space around an electric dipole unambiguously (Fig. 1). In the Feynman's Fig. 27-6 [5] (our Fig. 2) the Poynting vector is depicted (unambiguously) in space around an electric charge and a magnet (Feynman denotes the Poynting vector by **S**).

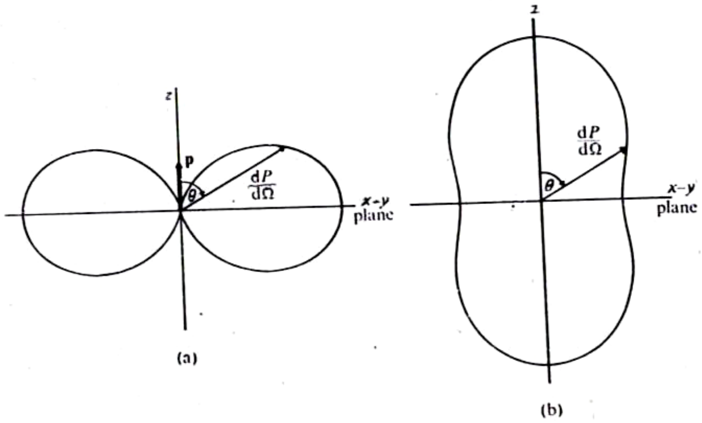

Fig.2.6. Angular distribution of electric dipole radiation for the case (a) of linear oscillation parallel to the z-axis, and (b) of circular motion in the x-y plane. The length of the radius vector is proportional to $\frac{dP}{d\Omega}$.

Fig. 1.

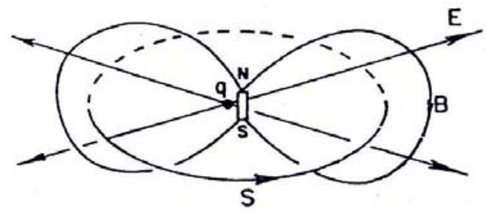

Fig. 27–6. A charge and a magnet produce a Poynting vector that circulates in closed loops.

Fig. 2.

It is important to examine the divergence of an energy-momentum tensor, $\partial_\mu T^{\lambda\mu}$. Let da_j be an element of a closed surface a which encloses a small volume V of a solid body. Then this volume acts on the rest part of the body by the force

$$F^i = \oint_{a=\partial V} T^{ij} da_j = \int_V \partial_j T^{ij} dV . \tag{1.4}$$

Thus, the divergence is the density of external forces acting on the material continuum:

$$f^i = dF^i / dV = \partial_j T^{ij} . \tag{1.5}$$

Let the material continuum be an electromagnetic field, which interacts with an electric 4-current j^μ. We know that an electromagnetic field acts on the 4-current by the Lorentz force density $f_L^\lambda = j_\mu F^{\lambda\mu}$ where $F^{\lambda\mu}$ is the field-strength tensor. Thus, the external 4-force density acting on the electromagnetic field is $f^\lambda = -j_\mu F^{\lambda\mu}$, and

$$-j_\mu F^{\lambda\mu} = \partial_\mu T^{\lambda\mu}, \quad j^\mu = \partial_\lambda F^{\lambda\mu} \tag{1.6}$$

where $T^{\lambda\mu}$ is the electrodynamics' energy-momentum tensor. Equation (1.6) is a key to identify this tensor $T^{\lambda\mu}$ in terms of the electromagnetic field.

As is known, the Maxwell energy-momentum tensor,

$$T^{\lambda\mu} = -g^{\lambda\alpha} F_{\alpha\nu} F^{\mu\nu} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta} / 4 , \tag{1.7}$$

obeys Equation (1.6). But the Equation admits an addition to the tensor, for example, $\partial_\alpha (A^\lambda F^{\mu\alpha})$, or $A^\lambda j^\mu$. Well! The Equation admits, but experiments admit no addition. Only the Maxwell tensor (1.7) gives the true 4-momentum dp^λ across a 3-target dV_μ. Only the Maxwell tensor provides the real pressure of light, Faraday's tensions, or the angular dependence of the energy flux density depicted by

Corney. The Feynman's Poynting vector $\mathbf{S}$ (Fig. 2) is a component of the Maxwell tensor. My radio receiver determines exactly this component of the Maxwell tensor (the Poynting vector). Thus, electrodynamics' energy-momentum tensor (1.7) is unambiguous.

2. What is the spin tensor?

Jan Weyssenhoff and A. Raabe [6]: "By spin-fluid we mean a fluid each element of which possesses besides energy and linear momentum also a certain amount of angular momentum, proportional – just as the energy and the linear momentum – to the volume of the element". In accordance with this sentence and analogically to (1.1) we define the 4-spin tensor $\Upsilon^{\lambda\mu\nu} = -\Upsilon^{\mu\lambda\nu}$ by the form:

$$\text{4-spin } ds^{\lambda\mu} \text{ across a 3-target } dV_\mu \text{ is } ds^{\lambda\mu} = \Upsilon^{\lambda\mu\nu} dV_\nu . \tag{2.1}$$

According to this definition, spin tensor is unambiguous as well as energy momentum tensor.

It is important to examine the divergence of a spin tensor, $\partial_\nu \Upsilon^{\lambda\mu\nu}$. Let dV_ν be an element of a closed 3-surface V which encloses a small 4-volume Ω of a material continuum with spin. Then this 4-volume supplies with the angular 4-momentum

$$J^{\lambda\mu} = \oint\limits_{V=\partial\Omega} (2r^{[\lambda}T^{\mu]\nu} + \Upsilon^{\lambda\mu\nu}) dV_\nu = \int\limits_{\Omega} (2T^{[\mu\lambda]} + 2r^{[\lambda} f^{\mu]} + \partial_\nu \Upsilon^{\lambda\mu\nu}) d\Omega \tag{2.2}$$

the rest part of the 4-continuum. If the external sources are absent, $J^{\lambda\mu} = 0,\ \ f^{\mu} = 0$,

$$2T^{[\lambda\mu]} = \partial_\nu \Upsilon^{\lambda\mu\nu} . \tag{2.3}$$

Since the Maxwell tensor (1.7) is symmetric, electrodynamics' spin tensor, if it exists, is divergence-free in a free field as well as electrodynamics' energy-momentum tensor in a free field.

3. The canonical formalism

Physicists tried to obtain electrodynamics' energy-momentum and spin tensors from the canonical Lagrangian [7 (4-111)], $\underset{c}{\mathsf{L}} = -F_{\mu\nu}F^{\mu\nu}/4$. By the Lagrange formalism, the canonical energy-momentum tensor [7 (4-113)]

$$\underset{c}{T}{}^{\lambda\mu} = \partial^\lambda A_\alpha \frac{\partial \underset{c}{\mathsf{L}}}{\partial(\partial_\mu A_\alpha)} - g^{\lambda\mu} \underset{c}{\mathsf{L}} = -\partial^\lambda A_\alpha F^{\mu\alpha} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta}/4 \tag{3.1}$$

and the canonical total angular momentum tensor [7 (4-147)]

$$\underset{c}{J}{}^{\lambda\mu\nu} = 2x^{[\lambda} \underset{c}{T}{}^{\mu]\nu} + \underset{c}{\Upsilon}{}^{\lambda\mu\nu} \tag{3.2}$$

are obtained. Here

$$\underset{c}{\Upsilon}{}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha} \frac{\partial \underset{c}{\mathsf{L}}}{\partial(\partial_\nu A_\alpha)} = -2A^{[\lambda} F^{\mu]\nu} , \tag{3.3}$$

is the canonical spin tensor [7 (4-150)]. Its space component is $\mathbf{E}\times\mathbf{A}$:

$$\underset{c}{\Upsilon}{}^{ij0} = \mathbf{E}\times\mathbf{A} . \tag{3.4}$$

Here the sense of a total angular momentum tensor, $J^{\lambda\mu\nu}$, is presented: $dJ^{\lambda\mu} = J^{\lambda\mu\nu}dV_\nu = 2x^{[\lambda}T^{\mu]\nu}dV_\nu + \Upsilon^{\lambda\mu\nu}dV_\nu$ is the total angular momentum in an element dV_ν, the corresponding integral is

$$J^{\lambda\mu} = L^{\lambda\mu} + S^{\lambda\mu} = \int_V 2x^{[\lambda}T^{\mu]\nu}dV_\nu + \int_V \Upsilon^{\lambda\mu\nu}dV_\nu . \tag{3.5}$$

It consists of two terms: the first term involves a moment of momentum and represents an orbital angular momentum; the second term is spin.

However, the canonical tensors (3.1), (3.2), (3.3) are not electrodynamics tensors. The canonical energy-momentum tensor has a wrong divergence. All these tensors obviously contradict experiments. For example, consider a uniform electric field:

$$A_0 = -Ex,\ \ A_x = 0,\ \ \partial_\alpha A^\alpha = 0,\ \ F_{x0} = -F^{x0} = \partial_x A_0 = -E , \tag{3.6}$$

where A_α is the magnetic vector potential from (3.1). The canonical energy density (3.1) is negative:

$$\underset{c}{T}^{00} = g^{00} F_{x0} F^{x0} / 2 = -E^2 / 2 . \tag{3.7}$$

Another example: consider a circularly polarized plane wave (or a central part of a corresponding light beam),

$$E^x = \cos(z-t),\ E^y = -\sin(z-t),\ B^x = \sin(z-t),\ B^y = \cos(z-t),\ A^x = \sin(z-t),\ A^y = \cos(z-t) \tag{3.8}$$

(for short we set $k = \omega = 1$). A calculation of components of the canonical spin tensor (6) yields

$$\underset{c}{\Upsilon}^{xy0} = 1,\quad \underset{c}{\Upsilon}^{xyz} = 1,\quad \underset{c}{\Upsilon}^{zxy} = A^x B_x = \sin^2(z-t),\quad \underset{c}{\Upsilon}^{yzx} = A^y B_y = \cos^2(z-t) . \tag{3.9}$$

This result is absurd because, though $\underset{c}{\Upsilon}^{xy0}$ and $\underset{c}{\Upsilon}^{xyz}$ are adequate, the result means that there are spin fluxes in y & x - directions, i.e. in the directions, which are transverse to the direction of the wave propagation.

An opinion exists that a change of the Lagrangian can help to obtain the Maxwell tensor (1.7) by the Lagrange formalism. A. Barut [8] presented a series of Lagrangians and field equations in Table 1.

Table I

Lagrangians and Equations of Motion for the Most Common Fields

Field	Lagrangian	Field Equations
Free Electromagnetic Field	$L_{\mathrm{I}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(E^2 - B^2)$ $L_{\mathrm{II}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(A^\mu{}_{,\mu})^2$ $L_{\mathrm{III}} = -\frac{1}{2}A^\mu{}_{,\nu}A_\mu{}^{,\nu}$ $L_{\mathrm{IV}} = \frac{1}{2}[A_\nu F^{\mu\nu}{}_{,\mu} - A_{\nu,\mu}F^{\mu\nu}] + \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$	$F^{\mu\nu}{}_{,\nu} = 0$ $\Box^2 A_\mu = 0$ $\Box^2 A_\mu = 0$ $\Box^2 A_\mu = 0$
Electromagnetic Field with an External Current	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\mu j^\mu$	$F^{\mu\nu}{}_{,\nu} = -\frac{1}{c}j^\mu$

However, A. Barut did not show energy-momentum and spin tensors corresponding to these Lagrangians. So, we add Table 2.

Table 2

Electrodynamics' Lagrangians, Energy-Momentum Tensors, and Spin Tensors

Lagrangian	Energy-momentum tensor	Spin tensor
$L_I = \underset{c}{L} = -F_{\mu\nu}F^{\mu\nu}/4$	$T_I^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} = -A_\nu{}^{,\lambda}F^{\mu\nu} + g^{\lambda\mu}F_{\sigma\nu}F^{\sigma\nu}/4$	$\Upsilon_I^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}$
$L_{II} = -F_{\mu\nu}F^{\mu\nu}/4 - (A^\mu{}_{,\mu})^2/2$	$T_{II}^{\lambda\mu} = T_I^{\lambda\mu} - A^{\mu,\lambda}A^\sigma{}_{,\sigma} + g^{\lambda\mu}(A^\sigma{}_{,\sigma})^2/2$	$\Upsilon_{II}^{\lambda\mu\nu} = \Upsilon_I^{\lambda\mu\nu} + 2A^{[\lambda}g^{\mu]\nu}A^\sigma{}_{,\sigma}$
$L_{III} = -A^\mu{}_{,\nu}A_\mu{}^{,\nu}/2$	$T_{III}^{\lambda\mu} = -A_\sigma{}^{,\lambda}A^{\sigma,\mu} + g^{\lambda\mu}A_{\sigma,\rho}A^{\sigma,\rho}$	$\Upsilon_{III}^{\lambda\mu\nu} = 2A^{[\lambda}A^{\mu],\nu}$
$L_V = -F_{\mu\nu}F^{\mu\nu}/4 - A_\sigma j^\sigma$	$T_V^{\lambda\mu} = T_I^{\lambda\mu} + g^{\lambda\mu}A_\sigma j^\sigma$	$\Upsilon_V^{\lambda\mu\nu} = \Upsilon_I^{\lambda\mu\nu}$

It is clear, none of these energy-momentum tensors is the Maxwell tensor. And what is more, none of these tensors differs from the Maxwell tensor by a divergence of an antisymmetric quantity. In other words, none of these tensors has true divergence (1.6). A method is unknown to get a tensor with the true divergence in the frame of the standard Lagrange formalism.

Nevertheless, physicists have created an illusion that the Maxwell tensor can be derived by so-called Belinfante-Rosenfeld procedure [9,10]. A specific terms,

$$\underset{st}{t}^{\lambda\mu} = -\partial_\nu \tilde{\Upsilon}^{\lambda\mu\nu}/2 = \partial_\nu A^\lambda F^{\mu\nu} + A^\lambda \partial_\nu F^{\mu\nu} \tag{3.10}$$

and

$$\underset{st}{m}^{\lambda\mu\nu} = -\partial_\kappa (x^{[\lambda}\tilde{\Upsilon}^{\mu]\nu\kappa}) , \tag{3.11}$$

are added to the canonical tensors (3.1) and (3.2) (here $\tilde{\Upsilon}^{\lambda\mu\nu} \overset{def}{=} \underset{c}{\Upsilon}^{\lambda\mu\nu} - \underset{c}{\Upsilon}^{\mu\nu\lambda} + \underset{c}{\Upsilon}^{\nu\lambda\mu} = -2A^\lambda F^{\mu\nu}$).

Unfortunately, this procedure does not give the Maxwell tensor. It gives strange tensors, energy-momentum $\underset{st}{T}{}^{\lambda\mu}$ and total angular momentum $\underset{st}{J}{}^{\lambda\mu\nu}$:

$$\underset{st}{T}{}^{\lambda\mu} = \underset{c}{T}{}^{\lambda\mu} + \underset{st}{t}{}^{\lambda\mu} = -\partial^{\lambda} A_{\nu} F^{\mu\nu} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta} / 4 + \partial_{\nu} (A^{\lambda} F^{\mu\nu}), \tag{3.12}$$

$$\underset{st}{J}{}^{\lambda\mu\nu} = \underset{c}{J}{}^{\lambda\mu\nu} + \underset{st}{m}{}^{\lambda\mu\nu} = \underset{c}{J}{}^{\lambda\mu\nu} + 2\partial_{\kappa} (x^{[\lambda} A^{\mu]} F^{\nu\kappa}). \tag{3.13}$$

The energy-momentum tensor $\underset{st}{T}{}^{\lambda\mu}$ (3.12) is obviously invalid, as well as the canonical energy-momentum tensor (3.1). So, the procedure [9,10] is unsuccessful, and the tensors (3.12), (3.13) are never used. But to make matter worse the procedure eliminates spin tensor at all:

$$\underset{st}{\Upsilon}{}^{\lambda\mu\nu} = \underset{st}{J}{}^{\lambda\mu\nu} - 2x^{[\lambda} \underset{st}{T}{}^{\mu]\nu} = \underset{c}{\Upsilon}{}^{\lambda\mu\nu} + \underset{st}{s}{}^{\lambda\mu\nu} = \underset{c}{\Upsilon}{}^{\lambda\mu\nu} + 2A^{[\lambda} F^{\mu]\nu} = 0; \tag{3.14}$$

here

$$\underset{st}{s}{}^{\lambda\mu\nu} = \underset{st}{m}{}^{\lambda\mu\nu} - 2x^{[\lambda} \underset{st}{t}{}^{\mu]\nu} = 2A^{[\lambda} F^{\mu]\nu} \tag{3.15}$$

is the Belinfante-Rosenfeld addend to the canonical spin tensor. As a result the form

$$J^{\lambda\mu} = \int_V 2x^{[\lambda} T^{\mu]\nu} dV_{\nu}, \tag{3.16}$$

instead of (3.5), is proclaimed as the total angular momentum, but $T^{\mu\nu}$ in (3.16) is the Maxwell tensor (1.7) instead of (3.12).

4. The Humblet transformation

Unfortunately, the form (3.16) raises the problem of electrodynamics' spin. In particular, spin of the circularly polarized light beam [11, problem 7.28]

$$\mathbf{E} = \exp(ikz - i\omega t)[\mathbf{x} + i\mathbf{y} + \frac{\mathbf{z}}{k}(i\partial_x - \partial_y)]E_0(x, y), \quad \mathbf{B} = -i\mathbf{E}/c \tag{4.1}$$

is not seen by the form (3.16). To see the spin, the Humblet transformation of form (3.16) is performed [12,13]:

$$\mathbf{J} = \varepsilon_0 \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) dV = \varepsilon_0 \int E^i (\mathbf{r} \times \nabla) A_i dV + \varepsilon_0 \int \mathbf{E} \times \mathbf{A} dV = \mathbf{L} + \mathbf{S}. \tag{4.2}$$

This transformation is presented as a decomposition of the moment of momentum $\mathbf{J}$ (3.16) into orbital and spin parts. However, the transformation returns us to the discarded formula (3.5). Moreover, the first term on the right, which is posed as the orbital part, obviously is zero for a symmetric beam [14]. Thus, this transformation claims that the total moment of momentum in such a beam is spin. Ohanian [13] writes, "this angular momentum is the spin of the wave".

In our opinion, it is illogical to consider the term $\varepsilon_0 \int \mathbf{E} \times \mathbf{A} dV$ of the decomposition as spin in the frame of the standard electrodynamics with Maxwell energy-momentum tensor and zero spin tensor because $\varepsilon_0 \mathbf{E} \times \mathbf{A}$ is a component of the canonical spin tensor (3.3) which is eliminated by the **Belinfante-Rosenfeld procedure** and so is absent in the theory. Besides there is a serious geometrical objection against the identification of the density of moment of momentum, $\varepsilon_0 \mathbf{r} \times (\mathbf{E} \times \mathbf{B})$, with the density $\varepsilon_0 \mathbf{E} \times \mathbf{A}$. The point is, $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ is localized on the surface of the beam [13]. A flow of circulating mass-energy flows there. On the contrary, $\mathbf{E} \times \mathbf{A}$ is distributed over the beam's body. Therefore a conclusion was made that integrating of quantity $\mathbf{E} \times \mathbf{A}$ is simply a method by which the moment of momentum of the circulating flow can be calculated and that the moment of momentum is an orbital angular momentum [14], while spin is apart from the expression $\varepsilon_0 \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) dV$.

Jackson [11] and Becker [15] agree that the term $\varepsilon_0 \mathbf{E} \times \mathbf{A}$ plays the role of a spin tensor and that the Humblet identity

$$\int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) dV = \int \mathbf{E} \times \mathbf{A} dV \tag{4.3}$$

identifies moment of momentum with spin. To confirm this identifying, Jackson [11, problem 7.27] and Becker [15, V. 2, p. 320] consider an electromagnetic radiation produced by a source localized in a finite region of space. They apply the Humblet transformation with the integration by parts to the radiation and obtain the same equality (4.2)

But they are mistaken! The equalities (4.2), (4.3) are invalid in this case because the integration by parts cannot be used when radiating into space. A straight calculation presented in Section 8 for the radiation of a rotating dipole gives

$$\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV = 2\int \mathbf{E}\times\mathbf{A}dV\,. \tag{4.4}$$

Somewhat such result must be expected if we attribute the sense of spin density to the integrand $\mathbf{E}\times\mathbf{A}$ because when radiating into space photons are variously directed, and their spins are not parallel to each other as in a beam. But, the point is, result (4.4) proves the moment of momentum, $\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV$, is not spin, $\int \mathbf{E}\times\mathbf{A}dV$!

Besides, as in the case of the beam, the quantities $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ and $\mathbf{E}\times\mathbf{A}$ are spatially separated in the case of the dipole radiation: moment of momentum, $\varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$, is radiated mainly near the plane of rotating of the dipole, while spin, $\varepsilon_0\mathbf{E}\times\mathbf{A}$, exists near the rotating axis, where the radiation is circularly or elliptically polarized [16].

There is one more important circumstance, which prevent the interpretation of the integral $\varepsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV$ as spin of a *radiation* irrespectively of the sense of $\mathbf{E}\times\mathbf{A}$. Vectors $\mathbf{E}$ and $\mathbf{B}$ of a radiation are perpendicular to the direction of the propagation i.e. $(\mathbf{E}\times\mathbf{B})\cdot\mathbf{k}=0$, where $\mathbf{k}$ is the wave vector. So $(\mathbf{r}\times(\mathbf{E}\times\mathbf{B}))\cdot\mathbf{k}=0$ for any radiation. Therefore the moment of momentum $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ must be calculated by the use of the non-radiative field, which is proportional to $1/r^2$ in the case of a radiation into space. This indicate non-radiative nature of the moment of momentum $\varepsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})dV$ while spin is an attribute of a radiation and must be calculated by the use of fields, which is proportional to $1/r$ only. Heitler, when defending spin nature of the moment of momentum, refers to a subtle interference effect on this subject [17]. But this explanation seems to be not convincing.

On our opinion, it is necessary to concede that $\varepsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ represents a moment of momentum, which has an orbital nature and does not represent spin of an electromagnetic radiation [18,19]. We use $\varepsilon_0\mathbf{E}\times\mathbf{A}$ as the spin density in Section 8, though spin density does not recognized in the modern electrodynamics.

5. Scheme of the calculations

There are two methods of calculations of energy, moment of momentum and spin fluxes of an electromagnetic radiation. These methods of course give identical results.

1. Volume density (of mass-energy or of moment of momentum) is integrated over a thin spherical layer (of thickness dr), which surrounds the source of the radiation, and then the integral is divided by dt, on the assumption $dr/dt=c$. So, the formulas for power of radiation and torque are obtained:

$$P=\int T^{tt}da_i dr^i/dt\,,\quad \tau^{ij}=\int 2r^{[i}T^{j]t}da_k dr^k/dt\ \ \text{[J]}. \tag{5.1}$$

Here components of the Maxwell energy-momentum tensor are used: T^{tt} is the volume density of mass-energy and T^{jt} is volume density of momentum, which is equal to the Poynting vector because of the symmetry of the Maxwell energy-momentum tensor,

$$T^{tt}=\varepsilon_0E^2/2+\mu_0B^2/2\,,\quad T^{jt}=T^{tj}=\varepsilon_0\mathbf{E}\times\mathbf{B}\ \ \text{[kg/m}^2\text{s]}. \tag{5.2}$$

2. However, it is more natural to integrate flux density components of the Maxwell tensor over a surface, which surrounds the source of the radiation:

$$P=\int T^{ti}da_i\ \ \text{[kg/s]},\quad \tau^{ij}=\int 2r^{[i}T^{j]k}da_k\,, \tag{5.3}$$

here T^{jk} is the Maxwell stress tensor.

The same two methods are applicable for a calculation of a spin radiation:

$$\underset{s}{\tau}^{ij} = \int \Upsilon^{ijt} da_k dr^k / dt \ \text{[J]}, \tag{5.4}$$

$$\underset{s}{\tau}^{ij} = \int \Upsilon^{ijk} da_k \,, \tag{5.5}$$

where $\Upsilon^{ijt} = \varepsilon_0 \mathbf{E} \times \mathbf{A}$, $\Upsilon^{\lambda\mu\nu}$ is a spin tensor.

We show by a straight calculation that, in the field of a rotating dipole, the ratio of power to moment of momentum flux,

$$P / \tau = \omega \,, \tag{5.6}$$

differs from the ratio of power to spin flux,

$$P / \underset{s}{\tau} = 2\omega \,, \tag{5.7}$$

in accordance with formula (4.4), and thus a moment of momentum is not spin. It is important that the ratio $P / \underset{s}{\tau}$ is the normal ratio of energy to spin for circularly polarized photons directed along the rotation axis $(\theta = 0)$,

$$\left\lfloor P / \underset{s}{\tau} \right\rfloor_{\theta=0} = \hbar\omega / \hbar = \omega \,, \tag{5.8}$$

rather than ratio (5.7).

We use the complex expressions for electromagnetic fields [15, V.1, p.284], [4, p.36],

$$\mathbf{E} = \left[\frac{\omega^2 (\mathbf{p}r^2 - (\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0 c^2 r^3} + \frac{i\omega(\mathbf{p}r^2 - 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 c r^4} - \frac{(\mathbf{p}r^2 - 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 r^5} \right] \exp(ikz - i\omega t) \tag{5.9}$$

$$\mathbf{H} = \left[\frac{\omega^2 \mathbf{r} \times \mathbf{p}}{4\pi c r^2} + \frac{i\omega \mathbf{r} \times \mathbf{p}}{4\pi r^3} \right] \exp(ikz - i\omega t) \tag{5.10}$$

The calculation of the power P by the method (5.3) is performed in [4, p.39] with a mistake. We give this calculation in Section 6, having corrected the mistake. The calculation of the moment of momentum flux τ^{xy} by the method (5.1) is performed in [4, p.41] with a mistake as well. We give this calculation in Section 7, having corrected the mistake. The calculation of the spin flux $\underset{s}{\tau}^{ij}$ by the method (5.4) is performed in Section 8.

6. Calculation of radiation power by method (5.3)

We integrate the Poynting vector T^{ti} (5.2) over a spherical surface of radius r:

$$P = \int T^{ti} da_i = \Re \int \varepsilon_0 \mu_0 (\mathbf{E} \times \overline{\mathbf{H}}) \mathbf{r} r d\Omega / 2 = \Re \int \mathbf{E} (\overline{\mathbf{H}} \times \mathbf{r}) r d\Omega / 2c^2 \,, \tag{6.1}$$

$d\Omega = \sin\theta d\theta d\varphi$, the line means complex conjugation. Substituting fields proportional to $1/r$ from (5.9), (5.10) yields

$$P = \int \frac{\omega^4 \,|\, \mathbf{p}r^2 - (\mathbf{pr})\mathbf{r} \,|^2}{32\pi^2 c^5 \varepsilon_0 r^4} d\Omega \,. \tag{6.2}$$

This expression coincides with formula (2.71) in [4]. Using Cartesian components of the single dipole rotating in x-y plain yields

$$p_x = \exp(-i\omega t), \ p_y = i \exp(-i\omega t) \ \text{[C m]}. \tag{6.3}$$

We obtain

$$\begin{aligned} &[\mathbf{p}r^2 - (\mathbf{pr})\mathbf{r}] \cdot [\overline{\mathbf{p}} r^2 - (\overline{\mathbf{p}}\mathbf{r})\mathbf{r}] / r^4 = \mathbf{p}\overline{\mathbf{p}} - (\mathbf{pr})(\overline{\mathbf{p}}\mathbf{r}) / r^2 = \\ &= p_x \overline{p}_x + p_y \overline{p}_y - (x + iy)(x - iy) / r^2 = 2 - \sin^2\theta = 1 + \cos^2\theta. \end{aligned} \tag{6.4}$$

So,

$$P = \int \frac{\omega^4 (1 + \cos^2\theta) \sin\theta}{32\pi^2 c^5 \varepsilon_0} d\theta d\varphi \,. \tag{6.5}$$

This result was obtained also as a solution of problem 1 in [20, § 67], but formula (2.73) in [4] inexplicably gives half of the quantity:

$$dP = \frac{\omega^4 (1+\cos^2\theta)}{64\pi^2 c^5 \varepsilon_0} d\Omega .$$

So, mass-energy flux in the field of a rotating dipole is

$$P = \frac{\omega^4}{6\pi c^5 \varepsilon_0} \text{ [kg/s]}. \tag{6.6}$$

This result is twice as much as the result [4, (2.74)].

7. Calculation of moment of momentum flux by method (5.1)

We integrate the moment of momentum volume density over a spherical layer

$$\tau^{ij} = \int 2r^{[i} T^{j]t} da_k dr^k / dt = \Re \int \varepsilon_0 \mu_0 \mathbf{r} \times (\mathbf{E} \times \overline{\mathbf{H}}) r^2 c d\Omega / 2 = \Re \int [\mathbf{E}(\mathbf{r}\overline{\mathbf{H}}) - (\mathbf{rE})\overline{\mathbf{H}}] r^2 d\Omega / 2c . \tag{7.1}$$

The first term on the right is zero, and the second term needs the use of the electromagnetic field, which is proportional to $1/r^2$

$$\tau^{ij} = \Re \int \mathbf{r} \frac{i\omega(-\mathbf{p}r^2 + 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 c r^4} \overline{\mathbf{H}} r^2 d\Omega / 2c = \Re \int \frac{i\omega 2(\mathbf{rp})}{4\pi\varepsilon_0 c r^2} \frac{\omega^2 \mathbf{r} \times \overline{\mathbf{p}}}{4\pi c^2} d\Omega / 2 = \Re \int \frac{i\omega^3 (\mathbf{rp}) \mathbf{r} \times \overline{\mathbf{p}}}{16\pi^2 \varepsilon_0 c^3 r^2} d\Omega . \tag{7.2}$$

This expression coincides with formula (2.78) in [3]. Since (6.3) we obtain

$$[(\mathbf{rp})\mathbf{r} \times \overline{\mathbf{p}}] / r^2 = [(xp_x + yp_y)(x\overline{p}_y - y\overline{p}_x)] / r^2 = -i(x^2 + y^2)/r^2 = -i \sin^2\theta . \tag{7.3}$$

As $\int_0^\pi \sin^3\theta d\theta = 4/3$, the torque emitted by the radiator is equal to

$$\tau^{xy} = \frac{\omega^3}{16\pi^2 \varepsilon_0 c^3} \int \sin^3\theta d\theta d\varphi = \frac{\omega^3}{6\pi\varepsilon_0 c^3} \text{ [J]}. \tag{7.4}$$

Contrary to (7.4), formula (2.80) in [4] inexplicably gives half of quantity (7.4), $\tau_z = \omega^3 / 12\pi^2 \varepsilon_0 c^3$.
However, somehow the ratio of power to moment of momentum flux is equal to frequency (5.6)

$$c^2 P / \tau = \omega . \tag{7.5}$$

8. Calculation of spin flux by method (5.4)

We integrate the spin volume density, $\Upsilon^{ijt} = \varepsilon_0 \mathbf{E} \times \mathbf{A}$, over a spherical layer

$$\underset{s}{\tau}^{xy} = \Re \int \varepsilon_0 E_{[x} \overline{A}_{y]} r^2 d\Omega dr / dt = \Re \int i\varepsilon_0 E_{[x} \overline{E}_{y]} r^2 c d\Omega / \omega . \tag{8.1}$$

By the use $\mathbf{E}$ from (2.9), which is proportional to $1/r$, and since (3.3) we obtain

$$\begin{aligned} E_{[x}\overline{E}_{y]} r^2 &= (E_x \overline{E}_y - E_y \overline{E}_x) r^2 / 2 = \\ &= \frac{\omega^4}{32\pi^2 \varepsilon_0^2 c^4 r^4} \{[r^2 - (x+iy)x][-ir^2 - (x-iy)y] - [ir^2 - (x+iy)y][r^2 - (x-iy)x]\} = \\ &= \frac{-i\omega^4}{16\pi^2 \varepsilon_0^2 c^4} \left(1 - \frac{x^2+y^2}{r^2}\right) = \frac{-i\omega^4}{16\pi^2 \varepsilon_0^2 c^4} (1 - \sin^2\theta) = \frac{-i\omega^4}{16\pi^2 \varepsilon_0^2 c^4} \cos^2\theta. \end{aligned} \tag{8.2}$$

As $\int_0^\pi \cos^2\theta \sin\theta d\theta = 2/3$, the spin flux emitted by the radiator is equal to

$$\underset{s}{\tau}^{xy} = \frac{\omega^3}{12\pi\varepsilon_0 c^3} \text{ [J]}. \tag{8.3}$$

This is half of the moment of momentum flux (7.4).

This result (8.3) was obtained by method (5.5) and by the use of spherical coordinates in paper [16]. In that paper, also the Corney's mistakes were indicated.

9. Conclusions, comments, and acknowledgements

A separate existence of spin and moment of momentum as different physical concepts is emphasized. These concepts originates in the Lagrange formalism with Noether's theorem where the canonical energy-momentum and spin tensor come into existence. We use the time component of the canonical spin tensor. We have ascertained the Humblet identity (4.3) for the light beam is accidental.

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