True energy-momentum tensors are unique. Electrodynamics spin tensor is not zero

Radi I. Khrapko*
Moscow Aviation Institute, 125993, Moscow, Russia

Abstract

A true energy-momentum tensor is unique and does not admit an addition of a term. The true electrodynamics' energy-momentum tensor is the Maxwell-Minkowski tensor. It cannot be got with the Lagrange formalism. The canonical energy-momentum and spin tensors are out of all relation to the physical reality. The true electrodynamics' spin tensor is not equal to a zero. So, electrodynamics' ponderomotive action comprises a force from the Maxwell stress tensor and a torque from the spin tensor.

A gauge non-invariant expression for the spin tensor is presented and is used for a consideration of a circularly polarized standing wave. A circularly polarized light beam carries a spin angular momentum and an orbital angular momentum. So, we double the beam's angular momentum. The Beth's experiment is considered.

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1 An outlook on the standard electrodynamics

The main points in the electrodynamics are an electric current four-vector density j^α and electromagnetic field which is described by a covariant skew-symmetric electromagnetic field tensor $F_{\mu\nu}$, or by a contravariant tensor $F^{\alpha\beta} = F_{\mu\nu}g^{\alpha\mu}g^{\beta\nu}$ ($\alpha,\ \nu, \ldots = 0,\ 1,\ 2,\ 3$, or $= t,\ x,\ y,\ z$). The four-vector density j^α comprises a charge density and an electric current three-vector density: $j^\alpha = \{j^0 = \rho,\ j^i\}$ ($i,\ k, \ldots = 1,\ 2,\ 3$, or $= x,\ y,\ z$). $g^{\alpha\mu} = \{g^{00} = 1,\ g^{ij} = -\delta^{ij}\}$.

We say that j^0 and j^i are *coordinates* of the vector j^α. We write coordinates in braces.

Instead of $F_{\mu\nu}$ and $F^{\alpha\beta}$ we use

$$B_{\mu\nu} \stackrel{def}{=} F_{\mu\nu} \quad \text{and} \quad H^{\alpha\beta} \stackrel{def}{=} F^{\alpha\beta}.$$

$B_{\mu\nu}$ comprises electric strength and magnetic induction; $H^{\alpha\beta}$ comprises electric induction and magnetic strength

$$B_{\mu\nu} = \{B_{i0} = E_i,\ B_{ij}\}; \qquad H^{\alpha\beta} = \{H^{0i} = D^i,\ H^{ij}\}$$

An interaction of the electric current and the electromagnetic field results in a four-force density $f_\mu = j^\nu B_{\mu\nu}$ comprising a power density and a Lorentz force density,

$$f_\mu = j^\nu B_{\mu\nu} = \{p = -j^k E_k,\ f_i = \rho E_i + j^k B_{ik}\}$$

*Email address: khrapko_ri@mai.ru

This force acts on the field. $-j^\nu B_{\mu\nu}$ acts on the current.

The second pair of Maxwell equations relates the contravariant electromagnetic tensor to the electric current,

$$j^\alpha = \partial_\beta H^{\alpha\beta},$$

while covariant electromagnetic tensor, which is a differential form, is closed,

$$\partial_{[\sigma} B_{\mu\nu]} = 0.$$

It is the first pair of Maxwell equations. The first pair is a consequence of an absence of a magnetic current $\xi_{\sigma\mu\nu}$,

$$3\partial_{[\sigma} B_{\mu\nu]} = \xi_{\sigma\mu\nu} = 0.$$

If the magnetic current is absent, the covariant electromagnetic tensor can be written as an exterior derivative of a covariant four-vector magnetic potential $\mathcal{A}_\nu$,

$$B_{\mu\nu} = 2\partial_{[\mu} \mathcal{A}_{\nu]}.$$

The four-vector $\mathcal{A}_\nu$ comprises a scalar potential and a covariant three-vector magnetic potential,:

$$\mathcal{A}_\nu = \{\mathcal{A}_0 = \phi,\ \mathcal{A}_j = A_j\}$$

So, $\mathcal{A}^i = -A^i$.

2 Energy-momentum tensor

The four-force density $f_\mu = j^\nu B_{\mu\nu}$ can be written as a divergence of a tensor T^α_μ (more precisely, tensor density),

$$f_\mu = j^\nu B_{\mu\nu} = \partial_\alpha T^\alpha_\mu.$$

The tensor T^α_μ is known as an energy-momentum tensor. Maxwell and Minkowski found this tensor from experimental data:

$$T^\alpha_\mu = -B_{\mu\nu} H^{\alpha\nu} + \delta^\alpha_\mu B_{\sigma\nu} H^{\sigma\nu}/4.$$

The contravariant Maxwell-Minkowski tensor is symmetric,

$$T^\alpha_\mu g^{\mu\beta} = T^{\beta\alpha} = T^{(\beta\alpha)}, \quad T^{[\beta\alpha]} = 0.$$

Now let us calculate a four-momentum which a field gets within a four-volume Ω,

$$P_\mu = \int_\Omega f_\mu d\Omega = \int_\Omega \partial_\alpha T^\alpha_\mu d\Omega.$$

By the Stokes theorem, we can write this integral as a surface integral over the supersurface of the four-volume,

$$P_\mu = \int_\Omega \partial_\alpha T^\alpha_\mu d\Omega = \oint_{\partial\Omega} T^\alpha_\mu dV_\alpha,$$

where dV_α is a three-element of the supersurface. So, if a field is bounded locally by an infinitesimal element dV_α, the element gets the infinitesimal four-momentum

$$dP_\mu = T^\alpha_\mu dV_\alpha.$$

This equation may be interpreted as a definition of an energy-momentum tensor. [1, 2]

Let dV_α is timelike, i.e. it contains the time axis. Let, for example, $dV_\alpha = \{dV_0 = 0,\ dV_i = da_i dt\}$, where da_i is a two-element of a surface. Then a four-force

$$dF_\mu = dP_\mu/dt = T^i_\mu da_i = \{d\mathcal{P} = T^i_0 da_i,\ dF_j = T^i_j da_i\}$$

comprises a power dP and a force dF_j which are associated with the element da_i. So,

$$T^i_0 = \mathcal{S}^i = -B_{0j}H^{ij} = E_j \epsilon^{ijk} H_k = \mathbf{E} \times \mathbf{H}$$

is the Poynting's vector, and T^i_j is the Maxwell stress tensor.

If dV_α is spacelike, $dV_\alpha = \{dV_0 = dV,\ dV_i = 0\}$, then a four-momentum

$$dP^\mu = T^{\mu 0} dV_0 = \{d\mathcal{E} = T^{00} dV,\ dP^j = T^{j0} dV\}$$

is associated with the volume element dV. So,

$$T^{j0} = -g^{ji} B_{ik} H^{0k} = \delta^{ji} \epsilon_{ikn} B^n D^k = \mathbf{D} \times \mathbf{B}$$

is the momentum density.

3 Canonical energy-momentum tensor

Apparently, the only theoretical way of getting Maxwell-Minkowski tensor is a variation of the canonical Lagrangian

$$\underset{c}{\Lambda} = -\frac{1}{4} B_{\mu\nu} H^{\mu\nu} \sqrt{-g}$$

with respect to the metric tensor in the Minkowski space [3, Sec. 94]. But we do not consider this way here. Within the scope of the standard Lagrange formalism the canonical Lagrangian gives a canonical energy-momentum tensor [3, Sec. 33],

$$\underset{c}{T}{}^\alpha_{\ \mu} = -H^{\alpha\nu} \partial_\mu \mathcal{A}_\nu + \delta^\alpha_\mu B_{\nu\sigma} H^{\nu\sigma}/4.$$

This tensor is "conserved", i.e.

$$\partial_\alpha \underset{c}{T}{}^\alpha_{\ \mu} = 0,$$

due to the uniformity of space-time according to the Noether theorem. But this tensor is out of all relation to the physical reality. It contradicts experience [2]. For example, in a constant uniform x directed magnetic field, $B^x = B, B^y = B^z = 0$,

$$B_{yz} = H^{yz} = B, \quad \mathcal{A}_y = -Bz/2, \quad \mathcal{A}_z = By/2,$$

the canonical tensor gives zero value of a field pressure across field lines:

$$\underset{c}{T}{}^y_{\ y} = \underset{c}{T}{}^z_{\ z} = 0,$$

what is wrong. Besides, the divergence of the canonical tensor is equal to a wrong expression

$$\partial_\alpha \underset{c}{T}{}^\alpha_{\ \mu} = j^\nu \partial_\mu \mathcal{A}_\nu.$$

Besides that, the contravariant canonical tensor is nonsymmetric,

$$\underset{c}{T}^{[\alpha\beta]} = -\partial_\mu(\mathcal{A}^{[\alpha}\partial^{\beta]}\mathcal{A}^\mu).$$

To turn the canonical energy-momentum tensor to Maxwell-Minkowski tensor theorists simply add two *ad hoc* terms [3, Sec. 33]:

$$\underset{c}{T}{}_\mu^{\ \alpha} + \partial_\nu(\mathcal{A}_\mu H^{\alpha\nu}) - \mathcal{A}_\mu j^\alpha = T_\mu^\alpha.$$

The second term repairs the divergence of the tensor, and than the first term symmetrizes the contravariant form of the canonical tensor. Certainly, this addition has no basis and is completely arbitrary. So, we have to realize that Lagrange formalism does not give a true energy-momentum tensor.

4 The uniqueness of a true energy-momentum tensor

Theorists simply ignore the second term of the addition, $-\mathcal{A}_\mu j^\alpha$. They do not see it. But they attempt to explain an addition of the first term, $\partial_\nu(\mathcal{A}_\mu H^{\alpha\nu})$. For example, Landau and Lifshitz wrote [3, Sec. 32],

"It is necessary to point out that the definition of the (energy-momentum) tensor $T^{\alpha\beta}$ is not unique. In fact, if $\underset{c}{T}^{\alpha\beta}$ is defined by

$$\underset{c}{T}{}_\mu^{\ \alpha} = q_{,\mu} \cdot \frac{\partial\Lambda}{\partial q_{,\alpha}} - \delta_\mu^\alpha \Lambda, \tag{32.3}$$

then any other tensor of the form

$$\underset{c}{T}{}^{\alpha\beta} + \frac{\partial}{\partial x^\gamma}\psi^{\alpha\beta\gamma}, \qquad \psi^{\alpha\beta\gamma} = -\psi^{\alpha\gamma\beta} \tag{32.7}$$

will also satisfy equation

$$\partial T^{\alpha\beta}/\partial x^\beta = 0, \tag{32.4}$$

since we have identically $\partial^2\psi^{\alpha\beta\gamma}/\partial x^\beta \partial x^\gamma = 0$. The total four-momentum of the system does not change, since ... we can write

$$\int \frac{\partial\psi^{\alpha\beta\gamma}}{\partial x^\gamma} dV_\beta = \frac{1}{2}\int \left(dV_\beta \frac{\partial\psi^{\alpha\beta\gamma}}{\partial x^\gamma} - dV_\gamma \frac{\partial\psi^{\alpha\beta\gamma}}{\partial x^\beta} \right) = \frac{1}{2}\oint \psi^{\alpha\beta\gamma} da_{\beta\gamma},$$

were the integral on the right side of the equation is extended over the (ordinary) surface which 'bond' the hypersurface over which the integration on the left is taken. This surface is clearly located at infinity in the three-dimensional space, and since neither field nor particles are present at infinity this integral is zero. Thus the fore-momentum of the system is, as it must be, a uniquely determined uantity."

Exactly the same explanation was given by L. H. Ryder [4, Sec. 3.2].

But it seems to be incorrect [1, 2].

$$\oint \psi^{\alpha\beta\gamma} da_{\beta\gamma} = 0$$

only if $\psi^{\alpha\beta\gamma}$ decreases on infinity rather quickly. We present here a three-dimensional analogy concerning an electric current I and its magnetic field H^{ij}:

$$\frac{1}{2}\oint H^{ij}dl_{ij} = \int \partial_j H^{ij} da_i = \int j^i da_i = I \neq 0.$$

So, the addition of $\partial_\gamma \psi^{\alpha\beta\gamma}$ can change the total 4-momentum of a system. For example, it is easy to express the energy-momentum tensor of an uniform ball of radius R in the form $\partial_\gamma \psi^{\alpha\beta\gamma}$:

$$\psi^{00i} = -\psi^{0i0} = \epsilon x^i/3 \quad (r < R), \quad \psi^{00i} = -\psi^{0i0} = \epsilon R^3 x^i/3r^3 \quad (r > R)$$

$$\text{give} \quad \partial_i \psi^{00i} = \epsilon \quad (r < R), \quad \partial_i \psi^{00i} = 0 \quad (r > R).$$

Obviously, an addition of $\partial_\gamma \psi^{\alpha\beta\gamma}$ can change the total 4-momentum and changes the medium locally.

5 Electrodynamics' spin tensor

So, the Lagrange formalism does not give a true energy-momentum tensor. But the formalism gives a more important thing. The formalism gives raise an idea of a classical spin. A spin tensor $\underset{c}{\Upsilon}^{\mu\nu\alpha}$, coupled with the energy-momentum tensor, arises due to the isotropy of space-time according to the Noether theorem. Essentially, we have a pair of the canonical tensors

$$\underset{c}{T}^{\mu\alpha} = -H^{\alpha\nu}\partial^\mu \mathcal{A}_\nu + g^{\mu\alpha} B_{\nu\sigma} H^{\nu\sigma}/4, \quad \underset{c}{\Upsilon}^{\mu\nu\alpha} = -2\mathcal{A}^{[\mu} H^{\nu]\alpha}.$$

The existence of the spin tensor imply that electromagnetic field acts on its boundary not only with the Maxwell stress tensor T^{ji} but also with a screw tensor Υ^{jki} (they are rather tensor densities). The stress tensor provides a force acting on a surface element, and the screw tensor provides a torque acting on the surface element,

$$dF^j = T^{ji} da_i, \qquad d\tau^{jk} = \Upsilon^{jki} da_i. \tag{1}$$

In Minkowski space we have

$$dP^\mu = T^{\mu\alpha} dV_\alpha, \qquad dS^{\mu\nu} = \Upsilon^{\mu\nu\alpha} dV_\alpha.$$

So, if a field is bounded locally by an infinitesimal element dV_α, the element gets the infinitesimal spin $dS^{\mu\nu}$.

It is natural that the canonical spin tensor is wrong just as the canonical energy-momentum tensor is. Indeed, let us consider, for example, a z directed circularly polarized plane wave:

$$\mathbf{E} = \mathbf{D} = (\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i\omega(t-z)}, \quad \mathbf{H} = \mathbf{B} = \omega \mathbf{A} = i\mathbf{E}.$$

The coordinate

$$\underset{c}{\Upsilon}^{zxy} = \mathcal{A}^x H^{zy} = A^x H_x$$

of the spin tensor is a current density of spin angular momentum about the y axis along the y axis. This quantity is not zero, what is not rightly.

Nevertheless a density of the spin about the z axis

$$\underset{c}{\Upsilon}^{jk0} = -2\mathcal{A}^{[j} H^{k]0} = -2A^{[j} D^{k]} = \mathbf{D} \times \mathbf{A},$$

and a current density of the spin about the z axis along the z axis

$$\underset{c}{\Upsilon}^{jkz} = -2\mathcal{A}^{[j}H^{k]z} = \mathbf{A}\cdot\mathbf{H}$$

correspond to reality for a plane wave (see Sec. 7).

Here a problem arises: what is an electrodynamics' true spin tensor. What must we add to the canonical spin tensor to get the true spin tensor?

Our answer is as follows: a spin addition, $\Delta\underset{c}{\Upsilon}^{\mu\nu\alpha}$, and the energy-momentum addition,

$$\Delta\underset{c}{T}^{\mu\alpha} = \partial_\nu(\mathcal{A}^\mu H^{\alpha\nu}) - \mathcal{A}^\mu j^\alpha,$$

must satisfy the equation,

$$\partial_\alpha(\Delta\underset{c}{\Upsilon}^{\mu\nu\alpha}) = 2\Delta\underset{c}{T}^{[\mu\nu]}. \qquad (2)$$

It is easy to find from (2) that

$$\Delta\underset{c}{\Upsilon}^{\mu\nu\alpha} = 2\mathcal{A}^{[\mu}H^{\nu]\alpha} + 2\mathcal{A}^{[\mu}\partial^{|\alpha|}\mathcal{A}^{\nu]},$$

and so we obtain [2]

$$\Upsilon^{\mu\nu\alpha} = \underset{c}{\Upsilon}^{\mu\nu\alpha} + \Delta\underset{c}{\Upsilon}^{\mu\nu\alpha} = 2\mathcal{A}^{[\mu}\partial^{|\alpha|}\mathcal{A}^{\nu]}.$$

Theorists recognize the equation (2), but they consider only the first term of $\Delta\underset{c}{T}^{\mu\alpha}$, that is $\partial_\nu(\mathcal{A}^\mu H^{\alpha\nu})$, as the energy-momentum addition. As a result, the equation (2) gives

$$\Delta\underset{c}{\tilde{\Upsilon}}^{\mu\nu\alpha} = 2\mathcal{A}^{[\mu}H^{\nu]\alpha}, \quad \text{and} \quad \tilde{\Upsilon}^{\mu\nu\alpha} = 0.$$

That is why a classical spin is absent in the modern electrodynamics.That is why they consider that a circularly polarized plane wave has no angular momentum.

So, our true spin tensor

$$\Upsilon^{\mu\nu\alpha} = 2\mathcal{A}^{[\mu}\partial^{|\alpha|}\mathcal{A}^{\nu]}.$$

is a function of the vector potential $\mathcal{A}_\mu$ and is not gauge invariant. We greet this fact. As is shown [5], $\mathcal{A}^\mu$ must satisfy the Lorentz condition, $\partial_\mu\mathcal{A}^\mu = 0$.

6 The plane wave problem

The elimination of the electrodynamics' spin tensor gives rise an opinion that total angular momentum J^{ik} is a moment of the linear momentum, i.e. the total angular momentum is an orbital angular momentum [6, 7, 8, 9, 10]

$$dJ^{ik} = dL^{ik} = 2r^{[i}dP^{k]}, \qquad (3)$$

where dP^k is proportional to Poynting's vector. This implies that a circularly polarized plane wave has no angular momentum directed along the propagation direction of the wave [11, 12, 6, 13, 14], that only a quasiplane wave of finite transverse extent, i.e. a beam, carries an angular momentum whose direction is along the propagation direction. In accord with (3) this angular momentum is provided by an outer region of the beam because a falloff in intensity gives rise E and B fields which are parallel to wave vector, and so the energy flow has components perpendicular to the wave vector [15]. They name this angular momentum *spin* [14]. Within an inner region of the beam E and B fields are perpendicular to the wave vector, and the mass-energy flow is parallel to the wave vector [7]. So, there is no angular momentum in the inner region [14].

To refute this paradigm I proposed a specific experiment in Oct., 1999. I proposed to consider a two-element flat target comprising an inner disc and a closely fitting outer annulus [16]. According to standard electrodynamics, the inner part of the target does not perceive a torque when the target absorbs a circularly polarized wave. But it is clear that really the disc does perceive a torque from the wave. The disc will be twisted in contradiction with the paradigm.

Allen and Padgett [17] agree with an incorrectness of the opinion that any plane wave has no angular momentum. But they try to endow a circularly polarized plane wave with angular momentum within the scope of the standard electrodynamics. The authors have attempted to explain the torque acting on the disc within the scope of the standard electrodynamics. They wrote, "Any form of aperture introduces an intensity gradient ... and a field component is induced in the propagation direction and so the dilemma is potentially resolved."

Alas! A small clearance between the inner disc and outer annulus does not aperture a wave and does not induce a field component in the propagation direction. The imaginary decomposition of the plane wave into three beams, the inner beam, the annular beam, and the remainder, is not capable to create longitudinal field components and, correspondingly, transverse momentum and a torque acting on the disc. Maxwell stress tensor cannot supply the disc with a torque.

To resolve the dilemma we must use the conception of classical electrodynamics' spin which is described by the spin tensor $\Upsilon^{\mu\nu\alpha}$. So, we must recognize that the standard classical electrodynamics is not complete. Electrodynamics' spin tensor is not zero [5], and "ponderomotive forces" acting on a surface element da_i consist of both, the force itself, and a torque (1).

So, the annulus of our target perceives the orbital angular momentum $L = \mathcal{E}/\omega$, the disc perceives a spin angular momentum $S = \mathcal{E}/\omega$, and the target as a whole perceives a total angular momentum

$$J = L + S = 2\mathcal{E}/\omega.$$

So, we double the beam's angular momentum.

This conclusion, naturally, must not conflict with the Beth's famous experiment [18]. And it is the case! It was found that Beth's half-wave plate perceives only spin angular momentum. The orbital angular momentum is eliminated by an interference of the passing and returning light beams in the experiment. Indeed, let us start from the Jackson's expression for a circularly polarized beam [7].

$$\mathbf{E}(x, y, z, t) = \Re\left\{[\hat{\mathbf{x}} + i\hat{\mathbf{y}} + \hat{\mathbf{z}}(i\partial_x - \partial_y)]E_0(x, y)e^{i(z-t)}\right\},$$

$$\mathbf{B}(x, y, z, t) = \Re\left\{[-\hat{i\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}(\partial_x + i\partial_y)]E_0(x, y)e^{i(z-t)}\right\}.$$

Here $E_0(x, y)$ is the electric field of the beam. $E_0(x, y) = \mathrm{Const}$ inside the beam, and $E_0(x, y) = 0$ outside the beam.

The returning light beam may be got by changing the sign of z and y. Adding up the passing and returning light beams we get interesting expressions,

$$\mathbf{E}_{\mathrm{tot}} = 2[E_0(\hat{\mathbf{x}}\cos z - \hat{\mathbf{y}}\sin z) - \hat{\mathbf{z}}(\sin z\partial_x E_0 + \cos z\partial_y E_0)]\cos t,$$

$$\mathbf{B}_{\mathrm{tot}} = -2[E_0(\hat{\mathbf{x}}\cos z - \hat{\mathbf{y}}\sin z) - \hat{\mathbf{z}}(\sin z\partial_x E_0 + \cos z\partial_y E_0)]\sin t.$$

The E and B fields are parallel everywhere. So, the Poynting vector is a zero.

7 The Humblet transformation

A considerable amount of papers is devoted to a circularly polarized light beam. They try to prove that the angular momentum (3) which is localized at the surface of the beam is distributed over the body of the beam and represents the spin of the beam.

A calculation of the orbital angular momentum (3),

$$L^{ij} = 2\int r^{[i}T^{j]0}dV,$$

for the Jackson's expression gives

$$L^z = \int E_0^2 dV/\omega, \qquad L^x = L^y = 0.$$

The energy of the field,

$$\mathcal{E} = \int T^{00}dV = \int E_0^2 dV = \omega L^z,$$

and the ratio $\mathcal{E}/L = \omega$ is the same as the ratio $\mathcal{E}/S$, i.e. energy/spin, for a photon. So, $L = S$ for the beam. But it does not follow that L *is* S. Simply, the total angular momentum of the beam, J, is twice the orbital angular momentum,

$$J^{ij} = L^{ij} + S^{ij} = 2\int r^{[i}T^{j]0}dV + \int \Upsilon^{ij0}dV.$$

Another method for the calculation of the orbital angular momentum of the beam was given by Humblet [12, 14]. He transforms angular momentum (3) into an integral of the coordinate $\underset{c}{\Upsilon}{}^{ij0}$,

$$\mathbf{L} = \int \mathbf{r}\times(\mathbf{D}\times\mathbf{B})dV \longrightarrow \int (\mathbf{D}\times\mathbf{A})dV.$$

Let us consider this transformation.

$$\begin{aligned} L^{ij} &= 2\int r^{[i}T^{j]0}dV = -2\int r^{[i}g^{j]k}B_{kl}H^{0l}dV = -4\int r^{[i}g^{j]k}\partial_{[k}\mathcal{A}_{l]}\cdot D^l dV \\ &= -2\int r^{[i}\partial^{j]}\mathcal{A}_l\cdot D^l dV + 2\int r^{[i}\partial_l(\mathcal{A}^{j]}D^l)dV. \end{aligned}$$

Ohanian [14] writes, "The first term in the Eq. represents the orbital angular momentum, and the second term the spin." But the derivative $\partial^j\mathcal{A}_l$ has only $j = z$ coordinate inside the beam, and, in any case, the term $\partial^j\mathcal{A}_l\cdot D^l$ is z directed. So, the first term is an integral moment of a longitudinal component of the momentum $T^{j0}dV$ and is equal to a zero if the origin of r^i is at the axis of symmetry. The first term bears no relation to the angular momentum of the beam.

The second term is transformed by an integration by parts,

$$2\int r^{[i}\partial_l(\mathcal{A}^{j]}D^l)dV = -2\int D^{[i}\mathcal{A}^{j]}dV = \int(\mathbf{D}\times\mathbf{A})dV.$$

And again we have $L = S$, but not L is S.

Vice versa. Since for any compact tensor field I^{ij}

$$\int I^{[ij]}dV = \int r^{[i}\partial_k I^{j]k}dV,$$

we have if

$$I^{ij} = \underset{c}{\Upsilon}{}^{ij0} = 2\mathcal{A}^{[i}D^{j]}, \quad S^{ij} = \int \underset{c}{\Upsilon}{}^{ij0}dV = +2\int r^{[i}\partial_k(\mathcal{A}^{j]}D^k)dV = 2\int r^{[i}T^{j]0}dV.$$

Ohanian [14] pay attention to a mathematical equivalence between the equation and calculating a magnetic moment, $P_{\rm m}^{ij}$, of a body as a moment of the Amperian magnetization current j^i,

$$P_{\rm m}^{ij} = \int I^{[ij]} dV = \int r^{[i}\partial_k I^{j]k} dV = \int r^{[i} j^{j]} dV,$$

where I^{ij} is the magnetization tensor, and $j^i = \partial_k I^{ik}$.

I am presenting another similar equation involving a magnetic strength H^{ij} and a current j^i,

$$\int H^{ij} dV = \int r^{[i}\partial_k H^{j]k} dV = \int r^{[i} j^{j]} dV,$$

But I think these relationships do not prove that a moment of a current and a magnetic strength, a moment of a current and a magnetization, a moment of a momentum and a spin have identical natures.

8 Circularly polarized standing wave

The electrodynamics is asymmetric. Magnetic induction is closed, but magnetic field strength has electric current as a source:

$$\partial_{[\alpha} B_{\beta\gamma]} = 0, \quad \partial_\nu H^{\mu\nu} = j^\mu.$$

So, a magnetic vector potential exists, but, generally speaking, an electric vector potential does not exist. However, when currents are absent the symmetry is restored, and a possibility to introduce an electric multivector potential $\Pi^{\mu\nu\sigma}$ appears. The electric multivector potential satisfies the equation

$$\partial_\sigma \Pi^{\mu\nu\sigma} = H^{\mu\nu}.$$

A covariant vector, dual relative to the three-vector potential,

$$\Pi_\alpha = \epsilon_{\alpha\mu\nu\sigma} \Pi^{\mu\nu\sigma},$$

is an analog of magnetic vector potential. We name it electric vector potential. In our case it satisfies the relationships:

$$\Pi_0 = 0, \quad \partial_0 \Pi_i = -H_i, \quad H_i = \epsilon_{ijk} H^{jk}.$$

The symmetry of the electrodynamics forces us to offer a symmetric expression for the spin tensor consisting of two parts, electric and magnetic.

$$\Upsilon^{\mu\nu\alpha} = \underset{e}{\Upsilon^{\mu\nu\alpha}} + \underset{m}{\Upsilon^{\mu\nu\alpha}} = A^{[\mu}\partial^{|\alpha|} A^{\nu]} + \Pi^{[\mu}\partial^{|\alpha|}\Pi^{\nu]}.$$

Here the symmetric expression is applied to a circularly polarized standing wave. We consider such a wave which falls upon a superconducting x, y-plain, and is reflected from it. The energy current density is equal to a zero in the wave, $T^{tz} = \mathbf{E} \times \mathbf{H} = 0$. But the electrical and magnetic energy densities vary with z in anti-phase. So the total energy density is constant. The momentum current density, i.e. the pressure, is constant too:

$$E^2/2 = 1 - \cos 2kz, \quad H^2/2 = 1 + \cos 2kz, \quad T^{tt} = T^{zz} = (E^2 + H^2)/2 = 2.$$

The spin current density must be zero, $\Upsilon^{xyz} = 0$, and it is expected that the spin density comprise electrical and magnetic parts which are shifted relative to one another. This result is obtained below.

A circularly polarized plane wave which propagates along z-direction involves the vectors $\mathbf{H}$, $\mathbf{E}$, $\mathbf{A}$, Π which lay in xy-plane, and we shall represent them by complex numbers instead of real parts of complex vectors. For example,

$$\mathbf{H} = \{H^x,\ H^y\} \to\ H = H^x + iH^y.$$

Then the product of a complex conjugate number $\overline{E}$ and other number H is expressed in terms of scalar and vector products of the corresponding vectors. For example,

$$\overline{E} \cdot H = (\mathbf{E} \cdot \mathbf{H}) + i(\mathbf{E} \times \mathbf{H})^z.$$

Since all this vectors do not vary with x and y, then

$$\mathrm{curl}\mathbf{H} = \{-\partial_z H^y,\ \partial_z H^x\} \to i\partial_z H, \quad \mathrm{curl}^{-1} \to -i\int dz.$$

The angular velocity of all the vectors is ω and the wave number along z-axis is $k = \omega$. Therefore

$$\mathbf{H} \to H_{01}e^{i\omega(t-z)} \text{ or, for a reflected wave, } H_{02}e^{i\omega(t+z)},$$

$$\partial_t \to i\omega, \quad \partial_z \to \mp i\omega, \quad \mathrm{curl} \to \pm\omega, \quad \mathrm{curl}^{-1} \to \pm 1/\omega.$$

If $z = 0$ at the superconducting x, y-plain, then the falling and reflected waves are recorded as

$$H_1 = e^{i\omega(t-z)}, \quad E_1 = -ie^{i\omega(t-z)}, \quad H_2 = e^{i\omega(t+z)}, \quad E_2 = ie^{i\omega(t+z)}.$$

The complex amplitudes are equal here: $H_{01} = H_{02} = 1,\ E_{01} = -i,\ E_{02} = i.$

Since $\mathbf{A} = \mathrm{curl}^{-1}\mathbf{H}$, $\Pi = \mathrm{curl}^{-1}\mathbf{E}$, the other complex amplitudes are received by a simple calculation (time derivative is designated by a point):

$$A_{01} = 1/\omega,\ \dot{A}_{01} = i,\ \Pi_{01} = -i/\omega,\ \dot{\Pi}_{01} = 1,\ A_{02} = -1/\omega,\ \dot{A}_{02} = -i,\ \Pi_{02} = -i/\omega,\ \dot{\Pi}_{02} = 1.$$

Now we calculate the electric and magnetic parts of the volumetric spin density.

$$\begin{aligned} \underset{e}{\Upsilon}^{xyt} &= (\mathbf{A} \times \dot{\mathbf{A}})/2 = \Im(\overline{(A_1 + A_2)} \cdot (\dot{A}_1 + \dot{A}_2))/2 \\ &= \Im((e^{-i\omega(t-z)} - e^{-i\omega(t+z)})i(e^{i\omega(t-z)} - e^{i\omega(t+z)}))/2\omega = (1 - \cos 2\omega z)/\omega, \end{aligned}$$

$$\underset{m}{\Upsilon}^{xyt} = (\Pi \times \dot{\Pi})/2 = \Im(\overline{(\Pi_1 + \Pi_2)} \cdot (\dot{\Pi}_1 + \dot{\Pi}_2))/2 = (1 + \cos 2\omega z)/\omega,$$

$$\Upsilon^{xyt} = \underset{e}{\Upsilon}^{xyt} + \underset{m}{\Upsilon}^{xyt} = 2/\omega.$$

So, the terms which oscillate along z-axis are canceled out. It is easy to calculate that the spin current density is equal to a zero (the prime denote the derivative with respect to z):

$$\underset{e}{\Upsilon}^{xyz} = -(\mathbf{A} \times \mathbf{A}')/2 = 0, \quad \underset{m}{\Upsilon}^{xyz} = -(\Pi \times \Pi')/2 = 0.$$

Notes

The material of this paper was rejected or ignored by the journals (dates of the submissions are in brackets): Phys. Rev. D (25 Sep 2001), Foundation of Physics (28 May 2001), American J. of Physics (15 Sep 1999, 10 Sep 2001, 28 Mar 2002), Acta Physica Polonica B (28 Jan 2002, 09 May 2002), Phys. Lett. A (22 July 2002), Experimental & Theor. Phys. Lett. (14 May 1998, 17 June 2002), J. Experimental & Theor. Phys. (27 Jan 1999, 25 Feb 1999, 13 Apr 2000, 25 May 2000, 16 May 2001, 26 Nov 2001), Theor. Math. Phys. (29 Apr 1999, 17 Feb 2000, 29 May 2000, 18 Oct 2000), Physics - Uspekhi (25 Feb 1999, 12 Jan 2000, 31 May 2000), Rus. Phys. J. (18 May 1999, 15 Oct 1999, 1 March 2000, 25 May 2000, 31 May 2001, 24 Nov 2001).

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