
# A rotating electric dipole radiates spin and orbital angular momentum

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# A rotating electric dipole radiates spin and orbital angular momentum

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According to the standard electrodynamics, a rotating electric dipole emits angular momentum mainly into the equatorial part of space situated near the plane of the rotation where polarization of the radiation is almost linear. Polar regions situated near the axis of rotating are scanty by the angular momentum, although they are intensively illuminated by the almost circularly polarized radiation, which carries spin angular momentum. A conclusion is made that the electrodynamics describes orbital angular momentum only and overlooks spin. This means that the electrodynamics is not complete. We use an electrodynamics' spin tensor and calculate the whole angular momentum flux radiated by the dipole.

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## 1. Introduction and conclusions

According to the standard electrodynamics [1, 2], a rotating electric dipole $p$ radiates time-average electromagnetic power[1]

$$P = dW/dt = \omega^4 p^2 / 6\pi \tag{1.1}$$

and angular momentum flux, i.e. torque[2]

$$\underset{L}{\tau} = dL/dt = \omega^3 p^2 / 6\pi \tag{1.2}$$

where $W$ and $L$ are the energy and angular momentum. Below we set $p = 1$, the speed of light $c = 1$, and $\varepsilon_0 = 1$.

The power (1.1) is usually obtained by integrating (see Sect. 2)

$$P = <\oint(\mathbf{E}\times\mathbf{B})\cdot\mathbf{da}> = \int \omega^4(\cos^2\theta + 1)\sin\theta\, d\theta d\varphi / 32\pi^2 = \omega^4/6\pi \tag{1.3}$$

where $\mathbf{E}\times\mathbf{B}$ is the Poynting vector and $\mathbf{da} = \hat{\mathbf{r}} r^2 \sin\theta\, d\theta d\varphi$ is a surface element ($\hat{\mathbf{r}} = \mathbf{r}/r$). However, the torque (1.2) is obtained not so trivially. Corney [2] integrates moment of the Poynting vector over a spherical layer,

$$d\mathbf{L}/dt = <\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})da>, \quad dL_z/dt = \int \omega^3 \sin^3\theta\, d\theta d\varphi / 16\pi^2 = \omega^3/6\pi. \tag{1.4}$$

The angular distribution of power (1.3),

$$dP/d\Omega = \omega^4(\cos^2\theta + 1)/32\pi^2, \tag{1.5}$$

is depicted in Fig. 1 from [2], and the angular distribution of the angular momentum flux relative to z-axis, according to (1.4),

$$dL_z/dtd\Omega = \omega^3 \sin^2/16\pi^2, \tag{1.6}$$

is depicted in Fig. 2. Here $d\Omega = \sin\theta\, d\theta d\varphi$.

The dipole radiation is elliptically polarized. The ratio of lengths of the half-axes equals

$$\cos\theta. \tag{1.7}$$

In particular, the $z$-directed ($\theta = 0$) radiation is circularly polarized, and the radiation in the equatorial plane ($\theta = \pi/2$) is linearly polarized. The degree of the circular polarization (1.7) is depicted in Fig. 3.

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[1] Corney [2] erroneously wrote that the power radiated by an electric dipole is $P = \omega^4 p^2 / 12\pi$ in both cases, in the case of circular oscillation and in the case of linear oscillation (p. 40). But his Fig.2.6(b) is correct.

[2] Corney [2] erroneously wrote $dL/dt = \omega^3 p^2 / 12\pi$. In reality, his eqn. (2.79) gives $dL/dt = \omega^3 p^2 / 6\pi$ because $p_x p_y^* - p_y p_x^* = \pm 2i$.

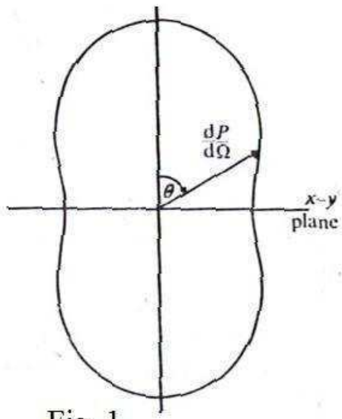


Fig. 1.
Angular distribution of the energy flux.
$dP/d\Omega \propto (\cos^2\theta + 1)$

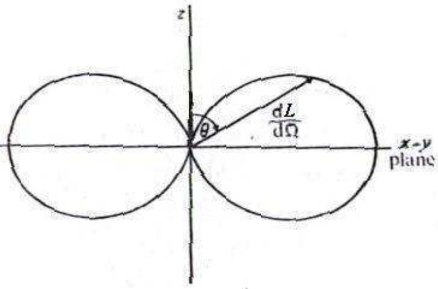


Fig. 2.
Angular distribution of z-component of the moment of momentum flux
$dL_z/dtd\Omega \propto \sin^2\theta$

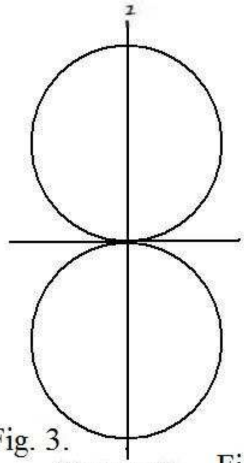


Fig. 3.
Degree of the circular polarization
$\sigma \cong \cos\theta$

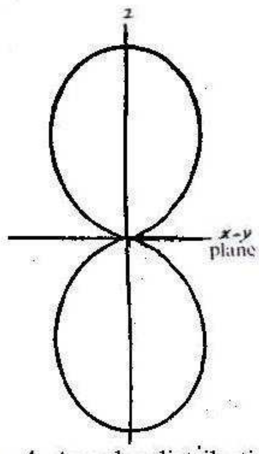


Fig. 4. Angular distribution of z-component of the spin flux
$dS_z/dtd\Omega \propto \cos^2\theta$

But, there is a puzzle here. According to Fig. 2, the angular momentum is emitted mainly into the equatorial part of space, situated near the plane of the rotation where the polarization is elliptic or linear, according to Fig. 3. Polar regions, situated near the *z*-axis, are scanty by the angular momentum, although they are intensively illuminated, according to Fig. 1, by the almost circularly polarized radiation.

However, R. Feynman, telling about spin of photons, clearly shows [3] that when a circularly polarized wave is absorbed, the absorbing medium gets spin angular momentum and energy in a $1/\omega$ ratio because a circularly polarized wave carries spin angular momentum.

From our viewpoint, this means that the angular momentum (1.2), (1.6) is an orbital angular momentum unconnected with spin of electromagnetic field. This angular momentum, possibly, has no wave nature because the Poynting vector does not need to have a wave nature. When rotation of a dipole is stationary, a torque acts on the dipole to compensate the radiated power (1.1)

$$P = \tau\omega, \tag{1.8}$$

and this torque is emitted into the equatorial region as orbital angular momentum flux (1.2).

From our viewpoint, the angular momentum (1.2), (1.6) does not exhaust the reality. Actually, the polar regions, illuminated by the circularly polarized light, get spin angular momentum. But calculating of this angular momentum calls for introducing a spin tensor into the standard electrodunamics.

Electron spins of material of the dipole may be sources of the spin radiation. The electron spins are gradually oriented in parallel to *z*-axis during the radiation. In other words, a rotating dipole is being magnetized in the transverse direction. A demagnetization of the dipole requires an additional torque applied to the dipole.

## 2. Calculation of the power and the orbital angular momentum

Here we detail eqns. (1.3) and (1.4). The **E** and **B** field satisfy equations [2, 4]:

$$4\pi E^i = 3p^k r_k r^i/r^5 - p^i/r^3 + 3\dot{p}^k r_k r^i/r^4 - \dot{p}^i/r^2 + \ddot{p}^k r_k r^i/r^3 - \ddot{p}^i/r, \tag{2.1}$$

$$4\pi B_{ik} = 2\dot{p}_{[i}r_{k]}/r^3 + 2\ddot{p}_{[i}r_{k]}/r^2. \tag{2.2}$$

We use spherical coordinate system $x^1 = r,\ x^2 = \theta,\ x^3 = \varphi$, with the metric

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \quad \sqrt{g} = r^2\sin\theta. \tag{2.3}$$

The unit dipole vector **p** has Cartesian components $p^x = \exp(-i\omega t),\ p^y = i\exp(-i\omega t),\ p^z = 0$, and spherical components:

$$p^i = \{p^r = \sin\theta,\ p^\theta = (\cos\theta)/r,\ p^\varphi = i/(r\sin\theta)\}\exp[i(\varphi - \omega t)] \tag{2.4}$$

$$p_i = \{p_r = \sin\theta,\ p_\theta = r\cos\theta,\ p_\varphi = ir\sin\theta\}\exp[i(\varphi - \omega t)] \tag{2.5}$$

The contravariant components of **E** and covariant components of **B** are

$$E^r = (2/r^3 - i2\omega/r^2)\sin\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi\,, \qquad (2.6)$$

$$E^\theta = (-1/r^4 + i\omega/r^3 + \omega^2/r^2)\cos\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi, \qquad (2.7)$$

$$E^\varphi = (-i/r^4 - \omega/r^3 + i\omega^2/r^2)\exp[i\varphi + i\omega(r-t)]/(4\pi\sin\theta), \qquad (2.8)$$

$$B_{r\theta} = (i\omega/r + \omega^2)\cos\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi, \qquad (2.9)$$

$$B_{\varphi r} = (\omega/r - i\omega^2)\sin\theta\,\exp[i\varphi + i\omega(r-t)]/4\pi, \quad B_{\theta\varphi} = 0. \qquad (2.10)$$

*r*-component of the Poynting vector, i.e. $T^{0r}$-component of the Maxwell tensor, is

$$T^{0r} = E^\theta B_{r\theta} - E^\varphi B_{\varphi r}\,. \qquad (2.11)$$

Using the higher powers of *r*, we obtain the time average quantity:

$$< T^{0r} > = \Re\{E^\theta B^*_{r\theta} - E^\varphi B^*_{\varphi r})/2 = \omega^4(\cos^2\theta + 1)/(32\pi^2 r^2) \qquad (2.12)$$

in accordance with (1.3).

For accurate calculating of the angular momentum flux density (1.4), (1.6), we must use components of the Maxwell stress tensor $T^{ij}$. $T^{ij}da_j = dF^i$ is the force acting on the surface element $da_j$, and $2r^{[k}T^{i]j}da_j = d\underset{L}{\tau}^{ki}$ is the torque acting on $da_j$. But the torque relative to *z*-axis is a three-vector $3\hat{z}^{[l}d\underset{L}{\tau}^{ki]}$, which must be dualized:

$$\hat{z}^l d\underset{L}{\tau}^{ki}\sqrt{g}\,e_{lki}/2 = d\underset{L}{\tau}_z\,. \qquad (2.13)$$

Here $\hat{z}^l$ is the unite *z*-coordinate vector and $\sqrt{g}\,e_{lki}$ is the antisymmetric tensor.

The component $T^{\varphi r}$ of the Maxwell tensor is

$$T^{\varphi r} = B_{r\theta}B^{\varphi\theta} - E_r E^\varphi = -E^r E^\varphi\,. \qquad (2.14)$$

The time average quantity is

$$< T^{\varphi r} > = \Re\{-E^r(E^\varphi)^*\}/2 = \omega^3/(16\pi^2 r^4)\,. \qquad (2.15)$$

The unit vector $\hat{\mathbf{z}}$ has spherical components

$$\hat{z}^r = \cos\theta, \;\; \hat{z}^\theta = -(\sin\theta)/r, \;\; \hat{z}^\varphi = 0 \qquad (2.16)$$

Using (2.13) yields (1.4) because $e_{\theta r\varphi} = -1$,

$$\underset{L}{\tau}_z = \oint \hat{z}^\theta r < T^{\varphi r} > \sqrt{g}\,e_{\theta r\varphi}da_r = \int \omega^3\sin^3\theta\, d\theta d\varphi/(16\pi^2) = \omega^3/(6\pi)\,. \qquad (2.17)$$

## 3. Radiation of spin

In this section we use an electromagnetic spin tensor [5-9]

$$\Upsilon^{\lambda\mu\nu} = \underset{e}{\Upsilon}^{\lambda\mu\nu} + \underset{m}{\Upsilon}^{\lambda\mu\nu} = A^{[\lambda}\nabla^{|\nu|}A^{\mu]} + \Pi^{[\lambda}\nabla^{|\nu|}\Pi^{\mu]}, \quad \lambda, \mu, \nu, ... = 0, 1, 2, 3\,, \qquad (3.1)$$

to calculate spin emited into the polar regions. Here $A^\lambda, \Pi^\lambda$ are the magnetic and electric vector potentials,

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}, \quad \Pi_\alpha = e_{\alpha\lambda\mu\nu}\Pi^{\lambda\mu\nu}, \quad \partial_\nu\Pi^{\lambda\mu\nu} = F^{\lambda\mu}\,. \qquad (3.2)$$

Because of spherical coordinates we use covariant derivatives in (3.1).

The sense of the spin tensor $\Upsilon^{ijk}$ is defined by the equation for a spin flux, $dS^{ij}/dt$, across the surface element $da_k$, i.e. for a spin torque on the element $da_k$,

$$\Upsilon^{ijk}da_k = dS^{ij}/dt = d\underset{S}{\tau}^{ij}\,. \qquad (3.3)$$

Now we calculate the spin radiation of the rotating dipole.

We set $A_0 = \phi = 0$. So, $A^i = -\int E^i dt = -iE^i/\omega$. Similarly, $\Pi^i = \int B^i dt = iB^i/\omega$, where

$$B^\theta = e^{\theta\varphi r}B_{\varphi r}/\sqrt{g} = (\omega/r^3 - i\omega^2/r^2)\,\exp[i\varphi + i\omega(r-t)]/4\pi, \qquad (3.4)$$

$$B^\varphi = e^{\varphi r\theta}B_{r\theta}/\sqrt{g} = (i\omega/r^3 + \omega^2/r^2)\cos\theta\,\exp[i\varphi + i\omega(r-t)]/(4\pi\sin\theta)\,. \qquad (3.5)$$

Therefore we have the time average spin tensor of the form

$$< \Upsilon^{ijk} = \underset{e}{\Upsilon}^{ijk} + \underset{m}{\Upsilon}^{ijk} > = \Re\{E^{*[i}\nabla^{|k|}E^{j]} + B^{*[i}\nabla^{|k|}B^{j]}\}/2\omega^2\,, \qquad (3.6)$$

Covariant derivatives, for example

$$\nabla_k E^i = \partial_k E^i + \Gamma^i_{jk} E^j , \qquad (3.7)$$

need connection coefficients $\Gamma^i_{jk}$ :

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cdot\cos\theta, \quad \Gamma^\varphi_{\theta\varphi} = \cos\theta/\sin\theta, \quad \Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = 1/r \qquad (3.8)$$

Using (2.6) – (2.8), (3.4) – (3.8) yields two components of the electric part of the spin tensor,

$$<\underset{e}{\Upsilon}^{\theta\varphi r}> = (\omega^3/r^4 - 2\omega/r^6)\cos\theta/(32\pi^2\sin\theta) , \qquad (3.9)$$

$$<\underset{e}{\Upsilon}^{\varphi rr}> = -\omega/(r^5 32\pi^2) , \qquad (3.10)$$

and the magnetic part

$$<\underset{m}{\Upsilon}^{\theta\varphi r}> = \omega^3\cos\theta/(r^4 32\pi^2\sin\theta) . \qquad (3.11)$$

So, we have two components of the spin tensor

$$<\Upsilon^{\theta\varphi r}> = <\underset{e}{\Upsilon}^{\theta\varphi r}> + <\underset{m}{\Upsilon}^{\theta\varphi r}> = (\omega^3/r^4 - \omega/r^6)\cos\theta/(16\pi^2\sin\theta) , \qquad (3.12)$$

$$<\Upsilon^{\varphi rr}> = <\underset{e}{\Upsilon}^{\varphi rr}> = -\omega/(r^5 32\pi^2) . \qquad (3.13)$$

The spin angular momentum flux relative to *z*-axis across an element $da_i$ is the dualized three-vector:

$$dS_z/dt = d\underset{S}{\tau}_z = \hat{z}^l \Upsilon^{ijk} da_k \sqrt{g}\, e_{lij}/2 = (\hat{z}^r \Upsilon^{\theta\varphi r} + \hat{z}^\theta \Upsilon^{\varphi rr}) da_r \sqrt{g} . \qquad (3.14)$$

as in the case of the angular momentum flux (2.13). Using (2.16) yields the time average spin flux radiated by the dipole

$$\underset{S}{\tau}_z = \int[\omega^3\cos^2\sin\theta + (\omega/2r^2)(-2\cos^2\theta\sin\theta + \sin^3\theta)]d\theta d\varphi/(16\pi^2) = \omega^3/(12\pi) . \qquad (3.15)$$

The second term in this integrand describes an interesting phenomenon. Except the part of the spin flux (3.15) that is radiated to infinity, a closed spin flow circulates not far from the rotating dipole. This spin flow is directed outside in the equatorial area, but is returned back in the polar area because

$$\int(-2\cos^2\theta\cdot\sin\theta + \sin^3\theta)d\theta d\varphi = 0 . \qquad (3.16)$$

This is a torque strength of the electromagnetic field.

Thus the circular oscillator radiates spin flux

$$\underset{S}{\tau}_z = dS_z/dt = \int\omega^3\cos^2\theta\sin\theta\, d\theta d\varphi/(16\pi^2) = \omega^3/(12\pi) . \qquad (3.17)$$

Angular distribution of this spin flux is

$$d\underset{S}{\tau}_z/d\Omega = \omega^3\cos^2\theta/(16\pi^2) \qquad (3.18)$$

instead of (1.6). This is depicted in Fig. 4. Note that the ratio of the spin flux density to the power density at $\theta = 0$ equals to $1/\omega$, just as for a photon, because the radiation is circularly polarized along the direction $\theta = 0$:

$$\left.\frac{\omega^3\cos^2\theta/(16\pi^2)}{\omega^4(\cos^2\theta+1)/(32\pi^2)}\right|_{\theta=0} = \frac{1}{\omega} . \qquad (3.19)$$

However, the total spin flux (3.17) is half of the total orbital angular momentum flux (1.2), (2.17).

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